All problems are required. Exam time is 3 hours. You have 10 points from the start - for taking the exam- (total: 100 pnts)

**Problem I**

1. Consider the polynomial with real coefficients \( f = x^3 - 2x^2 - 5x + 6 \).
   a) (3 pts) Calculate \( f(1) \);
   b) (3 pts) Determine the result and remainder of the division of \( f \) by \( x - 1 \);
   c) (4 pts) Solve the equation \( f(x) = 0 \).

2. (5 pts) Solve the equation:
   \[
   (\ln x)^3 - 2(\ln x)^2 - 5 \ln x + 6 = 0, \quad x > 0
   \] (1)

3. (10 pts) Determine \( x \in \mathbb{R} \) such that
   \[
   \int_0^x (3t^2 - 4t - 5)dt + 6 = 0
   \] (2)

4. In the cartesian coordinate system \( xOy \) consider the straight lines defined by the equations
   \[
   d_1 : 3x - 2y = 0, \quad d_2 : x + 3y - 11 = 0, \quad d_3 : 2x - 3y + 5 = 0
   \] (3)
   a) (3 pts) Determine the intersection point of the lines \( d_1 \) and \( d_2 \);
   b) (4 pts) Show that the lines \( d_1, d_2, d_3 \) intersect in the same point;
   c) (3 pts) Write down the equation of the circle of center \( O(0,0) \) which passes through the intersection point of the 3 lines.

**Problem II**

1. Consider the polynomial \( f = x^3 - x^2 + aX - 1, a \in \mathbb{R} \). For \( n \in \mathbb{N}^* \) denote \( S_n = x_1^n + x_2^n + x_3^n \), where \( x_1, x_2, x_3 \in \mathbb{C} \) are the roots of the polynomial \( f \).
   a) (6 pts) Show that \( S_3 - S_2 + aS_1 - 3 = 0 \).
   b) (6 pts) Determine \( a \in \mathbb{R} \) such that \( S_3 = 1 \).

2. For any \( x \in [0,1) \) define the sum:
   \[
   S_n(x) = \sqrt{1 - x} + x\sqrt{1 - x} + \ldots + x^{n-1}\sqrt{1 - x}, \forall n \in \mathbb{N}^*
   \] (4)
   a) (4 pts) Show that for any \( n \in \mathbb{N}^* \), we have:
   \[
   S_n(x) = \frac{1 - x^n}{1 - x}, \quad x \in [0,1)
   \] (5)
   b) (4 pts) Calculate \( \lim_{n \to \infty} S_n(x) \).
   c) (5 pts) Calculate \( \int_0^1 \sqrt{1 - x}dx \).

**Problem III**

In \( M_2(\mathbb{R}) \), the set of square matrices of order 2 over \( \mathbb{R} \), consider the matrix:

\[
X(a) = \begin{pmatrix}
1 + 5a & 10a \\
-2a & 1 - 4a
\end{pmatrix}
\] (6)
a) (3 pts) Calculate the determinant of the matrix \(X(a)\);

b) (3 pts) Show that for any \(a, b \in \mathbb{R}\), we have:

\[
X(a) \cdot X(b) = X(ab + a + b)
\]  

Consider the set \(G = \{X(a) | a \in (-1, \infty)\}\)

c) (3 pts) Show that \(G\) is a stable part of \(M_2(\mathbb{R})\) with respect to the operation of matrix multiplication;

d) (2 pts) Determine \((X(1))^2\).

e) (4 pts) Prove, using the method of mathematical induction, that for any \(n \in \mathbb{N}^*\), we have \((X(1))^n = X(2^n - 1)\).

**Problem IV**

Consider the function \(f : \mathbb{R} \to \mathbb{R}, f(x) = \cos x - 1 + x^2/2\).

a) (3 pts) Calculate \(\lim_{x \to 0} f(x)/x^4\).

b) (6 pts) Determine \(f'\) and \(f''\).

c) (2 pts) Show that \(f'(x) > 0, \forall x \in (0, \infty)\) and \(f'(x) < 0, \forall x \in (-\infty, 0)\).

d) (2 pts) Show that for any \(x \in \mathbb{R}, f(x) \geq 0\).

e) (2 pts) For the function \(g : \mathbb{R} \to \mathbb{R}, g(x) = \cos x\), show that the area of the plane surface delimited by the graph of the function, the \(Ox\) axis and the straight lines of equations \(x = 0, x = 1\), is larger than 5/6.