The Discretized String and a Finite realization of the Virasoro Algebra to O(1/N²)

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ABSTRACT: We consider the relativistic string as a system satisfying a set of defining constraints. For the continuous string this is fully equivalent to string theory derived from an Action principle. However in proceeding from the constraints, we find that the consideration of the string as a system of a finite (but large) set of degrees of freedom is feasible. From a rather straight forward analysis, we obtain a finite realization of the Virasoro Algebra to order 1/N² in addition to complete correspondence with closed string theory in the large N limit. Certain features of closed string theory are recovered and are shown to be independent of N, such features correspond to physical intuition.

1 Introduction

Much theoretical work in physics today focuses on the problem of finding a consistent formulation of non commutative geometry as a suitable background geometry for physical processes. The impetus for such work comes from three independent directions: The first being semi classical attempts at general relativity [1], The second being the desire for finite or regularizable field theories, and lastly from string theoretical considerations. Were spacetime to be quantized then, it makes sense to ask what consequences this might have for dynamical objects moving on it. Although we expect such scales to be far smaller than those at which we might have any dynamics, we might expect some manifestation of spatial graininess to be evident at scales a few orders of magnitude above that which determines the degree of non commutativity. It is with this
in mind that we undertake an analysis of the discretized relativistic string, or rather a system of $N$ degrees of freedom subject to a particular set of constraints that otherwise define the continuum relativistic string. We consider the closed string in particular as it simplifies the analysis tremendously in terms of not having to account for boundary effects. The results we obtain can easily be compared with the continuum situation (with only a few caveats) as $N$ tends to infinity, with complete correspondence. We come across some interesting results along the way, and thus isolate those aspects of perturbative string dynamics that depend on a continuum set of degrees of freedom, and those that don’t. In particular, naïve physical intuition is confirmed in that the lowest modes are almost entirely unaffected by the graininess (e.g. we still have a graviton), as is the case for a classical string composed of smaller entities, and higher modes are subject to a dispersion relation of the type familiar to us from classical strings. The key result from our perspective however, is a realization of the Virasoro Algebra up to terms of order $\frac{1}{N}$, which allows for the possibility of quantifiably accounting for non commutative effects in string theory. Specifically, we mean that studying deformations of the Virasoro Algebra might prove to be a natural manner in which to investigate the effects of non commuting spacetimes on String Theory. Before we begin then we would like to mention in passing that it may not be so unreasonable to consider a discretized version of the string, since considerations arising from M-Theory point to the fact that the string might be made up of still more fundamental entities.

2 The Discretized String

Classical and Quantum Field theory is often introduced to the physics student by means of the string, that is a collection of points subject to various restoring forces. The continuum limit is taken rather quickly as a means for a heuristic derivation of the various actions one encounters in field theory. The classical theory of a chain of masses (which would approximate to a string above a particular scale) without going to a continuum limit is well known (see [2] for
an excellent discussion) and has definite physical consequences, such as the dispersion relations for phonons in various solids (see [3]). However to treat a system of many particles subject to restoring forces relativistically would call for us having to abandon the Lagrangian formulation. This is because relativistic many body theory through an action principle is not very well defined, as can be seen from the fact that we would have "many times" corresponding to each degree of freedom. This is elegantly bypassed by the field formulation, and hence any tractable relativistic many body theory would have to be a field theory. However, considering the relativistic string as a system corresponding to a particular set of constraints allows us to bypass having to begin with an action principle (to which it is entirely equivalent) in the continuum, and thus allows us to consider the discretized situation without much difficulty. Let us see how this is so.

The entirety of perturbative closed string theory follows from enforcing the following constraints on a one dimensional extended object, subject to periodic boundary conditions [4]:

\[ P_\mu P^\mu + \frac{1}{2\pi} \partial_\sigma X_\mu X_{\mu,\sigma} = 0 \]
\[ P_\mu X^\mu_\sigma = 0 \]

Where in the continuum case \( P_\mu(\tau, \sigma) \) and \( X_\mu(\tau, \sigma) \) are the momentum and position fields respectively, and \( X_{\mu,\sigma} \) is the derivative of the position field with respect to the worldsheet parameter \( \sigma \). The normal way to proceed would be to consider the Fourier components of both constraints, and upon quantization, to consider the Hilbert space spanned by states annihilated by these constraints. We now do the same, but for a finite set of degrees of freedom. We will make frequent use of the formulae
\[
\sum_{n=-N}^{N-1} e^{i \pi (n-p) \xi} = N \delta_{kp} \\
\sum_{k=-N}^{N-1} e^{i \pi (k-n) \xi} = N \delta_{nm}
\]

Where we have deliberately chosen to expand symmetrically around the origin in sigma space for later convenience. In all that follows, \(Na = \pi, -\pi \leq na \leq \pi\), where \(a\) is the lattice spacing. This gives us the following expansion for the \(P^\mu\) and the \(X^\mu\):

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\[
p_k^\mu = \sqrt{\frac{\beta}{N}} \sum_{n=-N}^{N-1} e^{-i \frac{\pi n a_k}{N}} P_n^\mu \\
x_k^\mu = \sqrt{\frac{\beta}{N}} \sum_{n=-N}^{N-1} e^{-i \frac{\pi n a_k}{N}} X_n^\mu
\]

Where we denote the Fourier transformed quantities by lower case letters to avoid confusion. The subscripts on the position space coordinates denote the lattice site, and the subscript on the Fourier space coordinates denote the reciprocal lattice site. By reality of the coordinates, we have

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\[
p_{-k}^\mu = p_k^\mu \\
x_{-k}^\mu = x_k^\mu
\]

We chose to expand symmetrically around the origin to make the reality condition easier to impose. In momentum space, the \(\sigma\) derivatives of \(X^\mu\) become

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\[
(X^\mu)_m = \frac{1}{2\pi} (X^\mu_{m+1} - X^\mu_{m-1}) \\
= \sqrt{\frac{\beta}{N}} \sum_{k=-N}^{N-1} e^{i \frac{\pi n k a}{N}} x_k^\mu sin(\frac{\pi k}{N})
\]

or that

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\[(x^\mu_k)_k = \frac{1}{a} x^\mu_k \sin \left( \frac{\pi k}{N} \right)\]

When we rewrite (1) in terms of the Fourier modes, we get (Parseval’s theorem)

\[
\sum_{k = -N}^{N-1} p^\mu_k p^\mu_k + \frac{1}{2\pi a^2} \sin^2 \left( \frac{\pi k}{N} \right) x^\mu_k x^\mu_k = 0
\]

\[
\frac{i}{a} \sum_{k = -N}^{N-1} p^\mu_k x^\mu_k \sin \left( \frac{\pi k}{N} \right) = 0
\]

Now in the quantum theory, we realize that X and P are going to be hermitian operators. Since hermiticity is a basis independent property, we can say that the Fourier components \(x\) and \(p\) are also going to be hermitian operators. Thus we can rewrite the above as

\[
L_0 := \sum_{k = -N}^{N-1} p^\mu_k p^\mu_k + \frac{1}{2\pi a^2} \sin^2 \left( \frac{\pi k}{N} \right) x^\mu_k x^\mu_k = 0
\]

\[
\frac{i}{a} \sum_{k = -N}^{N-1} p^\mu_k x^\mu_k \sin \left( \frac{\pi k}{N} \right) = 0
\]

Now, taking the Fourier components of these constraints (that is multiplying by \(e^{-i(k+p)x/a}\) and summing over lattice sites \(m\)), we get in addition the following:

\[
L_p := \sum_{k = -N}^{N-1} p^\mu_{k+p} p^\mu_{k+p} + \frac{1}{2\pi a^2} \sin \left( \frac{\pi k}{N} \right) \frac{1}{2\pi a} \sin \left( \frac{\pi k}{N} \right) x^\mu_{k+p} x^\mu_{k+p} = 0
\]

\[
\frac{i}{a} \sum_{k = -N}^{N-1} p^\mu_{k+p} x^\mu_{k+p} \sin \left( \frac{\pi k}{N} \right) = 0
\]

Now, let us proceed as usual and write down

\[
a^{\mu}_k = \frac{1}{\sqrt{2}} \left( x^\mu_k e^{i\frac{\pi k}{2N}} \sin \frac{\pi k}{N} + ip^\mu_k \right)
\]

\[
a^{\mu \dagger}_k = \frac{1}{\sqrt{2}} \left( x^\mu_k e^{-i\frac{\pi k}{2N}} \sin \frac{\pi k}{N} - ip^\mu_k \right)
\]

Which, upon imposing the canonical commutation relations on the Fourier components of X and P (again we use the fact that commutation relations are basis independent), we get
\[ [a_k^\mu, a_k^{\nu\dagger}] = -\eta_{\mu\nu} \delta_{kk} \Omega(k) \]

where \( \Omega(k) = \frac{1}{\sqrt{2\pi a}} \sin \frac{\pi k}{N} \). Now we immediately see that \( \Omega(k) \) is negative when \( k \) is negative. All this means is that we have to switch what we mean by a creation operator and an annihilation operator in order to have a well defined theory. That is we impose \( a_k|0\rangle = 0 \) for \( k > 0 \) and \( a_k^\dagger|0\rangle = 0 \) for \( k < 0 \), and then we interchange the adjoint label on all operators for \( k < 0 \) (since in either case, one is the adjoint of the other). We assume this to be done already, and all this can be effected in one step if we take the absolute value of \( \Omega(k) \) whenever we see it appear in our formulae. In the traditional approach to closed string theory, we divide our oscillator operators into two bunches and consider each separately. We are accounting for this in a rather roundabout way here.

Now we can proceed to rewrite the constraints (7) and (8) in terms of these operators. We see that the second constraint of each (7) and (8) serves to eliminate cross terms, leaving us with

\[
L_0 = \sum a_k^\mu a_{\mu k} + a_k^\dagger a_k^{\mu\dagger} \\
L_p = 2 \sum a_k^\mu a_{\mu k+p} + a_k^\dagger a_k^{\mu+p\dagger}
\]

Now normal ordering for \( L_0 \) gives us

\[
L_0 = 2 \sum a_k^\mu a_{\mu k} - [a_k^\mu, a_k^{\mu\dagger}] \\
= 2 \sum a_k^\mu a_{\mu k} - D \sum |\Omega(k)|
\]

In closed string theory, the existence of tachyons trace their origins to this sum over modes. We can derive an explicit formula for this sum as follows:
\[ D \sum_{k = -N}^{N-1} |\Omega(k)| \]

\[ = 2D \sum_{k = 0}^{N-1} |\Omega(k)| \]

\[ = \frac{2D}{\sqrt{2\pi a}} \sum_{k = 0}^{N-1} \sin \frac{\pi k}{N} \]

and using the formula \( \sum_{k = 0}^{N-1} x^k = \frac{1-x^n}{1-x} \) we get

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\[ D \sum_{k = -N}^{N-1} |\Omega(k)| = \frac{2D}{\sqrt{2\pi a}} \cot \frac{\pi}{2N} \]

\[ = \frac{2D}{\sqrt{2\pi a}} \cot \frac{\pi}{2} \]

Now, we realize that this formula diverges as \( a \) tends to zero, but we can compute the part that stays finite, and throw away the infinite contribution (this is essentially what we do when we \( \zeta \) function regularize the infinite sum in continuum string theory). Leaving us with the contribution

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\[ D \sum_{k = -N}^{N-1} |\Omega(k)| = -\frac{D}{\sqrt{2\pi a}} \]

Where in fact, we should really have \( D - 2 \) in place of \( D \) since this is the number of independent oscillators (or if you like once we have eliminated all our gauge degrees of freedom, we would only have \( D - 2 \) degrees of freedom in our problem). We will see further on that we are actually measuring energy in units of \( \sqrt{\frac{1}{2\pi}} \) and so we note here that we will have a massless state of some description in our theory only is \( D - 2 \) is divisible by three. Thus the possible dimensionalities of theories which admit massless particles are 5, 8, 11...26 etc. So we see that the magic number 26 is consistent with our treatment, though at this stage of the analysis this is far from inevitable. So we can now rewrite our constraints as follows

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\[ L_0 = 2 \sum_{k = -N}^{N-1} g_{\mu k}^+ a_{\mu k} + \frac{D-2}{\sqrt{2\pi a}} \]

\[ L_P = 2 \sum_{k = -N}^{N-1} g_{\mu k}^+ a_{\mu k+P} \]
If we realize that the zero modes correspond to center of mass motion, we can rewrite the above as

\[ L_0 = 2P^\mu P_\mu + 2 \sum \delta_k a_k^{\mu+} a_{\mu k} + \frac{D-2}{2\varepsilon} \]
\[ L_p = 2P^\mu a_{\mu p} + 2 \sum \delta_k a_k^{\mu+} a_{\mu k+p} \]

And if we impose that any states in our Hilbert space be annihilated by these operators, we can construct the particle content of our theory. Since \( \frac{L_0}{\varepsilon} \mid_{\text{state}} = 0 \) if \( L_p \mid_{\text{state}} = 0 \), we can consider the rescaled operators \( l_p = \frac{L_p}{\varepsilon} \). However, we should first check the algebra satisfied by these operators. It is a straightforward matter to check that

\[ [l_m, l_n] = \sum_l a_l^{\mu+} a_{\mu l+m+n} [\Omega(l + m) - \Omega(l + n)] \]

Thus the algebra does not appear to close. However, if we recall our definition of \( \Omega(k) \) and consider what the expression above might be in the limiting case, we find that

\[ \Omega(l + m) - \Omega(l + n) = \frac{1}{\sqrt{2\pi}} \left[ \sin \frac{\pi(l + m)}{N} - \sin \frac{\pi(l + n)}{N} \right] \]

Now we perform a \( \frac{1}{N} \) expansion, and realizing that \( Na = \pi \), we get that

\[ \Omega(l + m) - \Omega(l + n) = \frac{1}{\sqrt{2\pi}} \left[ \frac{\pi(l + m)}{N} - \frac{\pi^2(l + m)^2}{6N^2} + \frac{\pi^3(l + m)^3}{6N^3} - \frac{\pi(l + n)}{N} \right] \]
\[ = \frac{1}{\sqrt{2\pi}} \left[ (l + m) - (l + n) - \frac{\pi^2}{6N^2} (l + m)^3 - (l + n)^3 \right] \]
\[ = \frac{1}{\sqrt{2\pi}} (m - n) + O\left(\frac{1}{N^2}\right) \]

Thus we get

\[ 20 \]
\[ [l_m, l_n] = \frac{(m-n)}{2\theta} l_{m+n} + O(\frac{1}{\sqrt{\theta}}) \]

Which after a further rescaling will be equivalent to the Virasoro Algebra of the continuum string up to order \( \frac{1}{\sqrt{\theta}} \). Thus we see that in our treatment of the discretized string, we recover a central feature of the continuum string up to a quantifiable departure. We can also take this as a starting point of our investigation, and consider deformations of the Virasoro Algebra as a means to study the effects of non commuting spacetimes on string theory.

We now focus on the particle content of this theory. Again, our treatment is standard and is a direct application of the method used in continuum string theory [4]. We thus impose that physical states be annihilated by the constraints \( l_p \), namely

\[
[P^\mu P_\mu + \sum_k \omega_k a^\mu_k a_{\mu k} + \frac{D-2}{\sqrt{2\pi g_1}}]_{\text{state}} = 0
\]

\[
[P^\mu a_\mu + \sum_k \omega_k a^\mu_k a_{\mu k+p}]_{\text{state}} = 0
\]

Thus the masses of the physical particles are given by

\[
m^2 = -\sum_k \omega_k \Omega(k) a^\mu_k a_{\mu k+p} - \frac{D-2}{\sqrt{2g_1}}
\]

where

\[
[a^\mu_k, a^\nu_l] = -\eta^{\mu\nu} \delta_{kl}
\]

Considering the effect of this operator on \( a^\nu_k |0\rangle \), we get
\[- \sum_{k \neq 0} \Omega(k) \alpha_k^\dagger \alpha_{p+k} - \frac{D-2}{2\pi^2} \alpha_k^\dagger |0\rangle = \left[ \Omega(k) - \frac{D-2}{2\pi^2} \right] \alpha_k^\dagger |0\rangle\]

Thus the masses squared of our particles is given by (substituting in D=26)

\[m^2 = [\Omega(k) - \frac{4}{\sqrt{2\pi}}]\]

We can evaluate this rather easily. First we note that for each energy, we have two independent oscillators, since \(\sin(x) = \sin(\pi - x)\). Or rather, as one would expect in closed string theory, particles of such masses are obtained by taking tensor products of both operators. Noting that we have for k large or small enough,

\[\Omega(k) = \Omega(N - k) = \frac{1}{\sqrt{2\pi a}} \sin(\pi - \frac{k\pi}{N})\]

\[= \frac{1}{\sqrt{2\pi a}} \left[ \sin(\pi) \cos(\frac{k\pi}{N}) - \cos(\pi) \sin(\frac{k\pi}{N}) \right]\]

\[= \frac{1}{\sqrt{2\pi a}} \sin(\frac{k\pi}{N})\]

\[= \frac{1}{\sqrt{2\pi a}} \frac{k\pi}{N}\]

\[= \frac{\frac{4}{\sqrt{2\pi}}}{\sqrt{2\pi}}\]

And so, when we apply (22) to a state of the form \(\alpha_k^\dagger \alpha_{N-k}^\dagger |0\rangle\), we get

\[m^2 = \frac{1}{\sqrt{2\pi}} (2k - 4)\]

We note that (27) is independent of the magnitude of \(N\), a feature one might anticipate from physical intuition since we expect the lowest modes to be independent of any small scale detail. We again make contact with continuum closed string theory in that we have one tachyon in our theory, corresponding to \(k = 1\), and one massless particle in our theory corresponding to \(k = 2\) (\(k = 0\) of course being the c.o.m. mode). Thus we recover the low lying spectrum exactly, without recourse to any limits.
Upon further examination of this massless particle, $h_{\mu \nu} \alpha_2^{\mu} \alpha_2^{\nu} [0] = 0$ that is, imposing the constraints $L_0 h_{\mu \nu} \alpha_2^{\mu} \alpha_2^{\nu} [0] = 0$ we find that $h_{\mu \nu}$ must be a traceless tensor such that $P^\mu h_\mu^\nu = 0$ where $P^\mu$ is the center of mass 4 momentum. Thus our massless particle has the defining properties of the graviton. Thus we see that yet another central feature of continuum closed string theory is recovered in our model.

3 Concluding Remarks

In the investigation just concluded, we speculated on the nature of any manifestations spatial graininess might have on string theory. We found that the rather remarkable set of constraints (1) which serve to define continuum string theory, also permit a study of the discretized string which was not possible before. We find that we recover almost entirely the key aspects of continuum string theory, in particular the nature of the lowest lying states, which is something physical intuition might suggest. The key results is that such a system satisfies the Virasoro Algebra almost exactly, with departures quantifiable in a $1/k$ expansion. Thus the study of deformed Virasoro Algebras might serve as a starting point in any serious study of non commutative string theory.

4 References

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