Lecture 7 p 4

The curvature is a scalar. To find the curvature, we're going to consider a 2-D section of the space.

(Why? well, when there's only one curvature we can pick any two dimensions, curvature only defined in Dim ≥ 2.)

Let's look at the section \( \Theta = \text{constant} = \frac{\pi}{2} \).

The spatial part of the metric is then
\[
G(r) \, dr^2 + r^2 d\phi^2 \quad (\sin^2 \Theta = 1 \, \text{if} \, \Theta = \frac{\pi}{2})
\]

\[
G_{\phi\phi} = \begin{pmatrix}
G(r) & 0 \\
0 & r^2
\end{pmatrix}
\]

In 2-D, the curvature can either be derived from the definition of \( R_{\mu\nu} \) or, from a formula derived from Gauss:

\[
K = \frac{1}{2 \, G(r) r^2} \left\{ 0 - 2 + \frac{1}{2 \, G(r)} \left[ G'(r) r^2 + \frac{1}{2} \left( \frac{G'}{G} \right)^2 \right] \right\}
\]

\[
K = \frac{1}{2 \, G(r) r^2} \left\{ -2 + \frac{G'(r)}{G(r)} + 2 \right\}
\]

\[
K = \frac{G'(r)}{2 \, G(r) r^2} \rightarrow \Delta \text{K} = \frac{G'(r)}{(G(r))^2} = \frac{3}{r} \left( -\frac{1}{G(r)} \right)
\]

\[
\frac{1}{G(r)} = C - \frac{2 \, K}{G(r)}
\]

\[
G(r) = \frac{1}{C - K r^2} \rightarrow C = 0 \quad \text{well, if} \, K = 0 \, \text{space metric has} \, q(t) = \frac{1}{3}
\]
Therefore 
\[ S(r) = \frac{1}{1 - Kr^2} \]

so 
\[ \sum \left( \frac{c^2 dt^2}{1 - Kr^2} - \frac{dr^2}{r^2} - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \right) \]

\[ \text{Note } Kr^2 = \text{dimensionless} \quad \Rightarrow K \text{ has dimensions } \frac{1}{r^2} \text{ (i.e. sphere)} \]

However, we haven't finished: although the cosmological principle is satisfied, in general the perfect cosmological principle is not 
\[ r \to r(t) \]

Let's define \( r = a(t) x \quad \Rightarrow \frac{dr}{dt} = a(t) \frac{dx}{dt} \]

\[ ds^2 = c^2 dt^2 - a^2(t) \left[ \frac{dx^2}{1 - ka^2 x^2} - a^2 x^2 d\theta^2 - a^2 x^2 \sin^2 \theta d\phi^2 \right] \]

\[ k = \frac{ka^2}{1 - ka^2 x^2} \quad \text{dimensionless curvature} \]

\[ ds^2 = c^2 dt^2 - a^2(t) \left[ \frac{dx^2}{1 - k x^2} + x^2 d\theta^2 + x^2 \sin^2 \theta d\phi^2 \right] \]

This (and from now on we'll use \( r \) instead of \( x \) as a comoving coordinate) is our metric tensor; the Robertson-Walker metric.

Because the stress-energy tensor has only two non-trivial components, so the Einstein tensors must also have two non-trivial components \( G_{00} \) and \( G_{\mu \nu} \) \((\mu=0)\).

Now that we have the metric, we can compute these components directly from \( g_{\mu \nu} \):
8.1a
Nonline element

From $g_{\mu \nu}$ our only non-zero elements are

\[ g_{00} = e^2 \quad g_{11} = -a^2 \quad g_{22} = r^2 a^2 \quad g_{33} = -\alpha \gamma^{r_0} \sin^2 \theta \]

Note that because \( g^{\mu \nu} g_{\mu \nu} = 1 \)

\[ g^{00} = e^2 \quad g^{11} = -\frac{1}{a^2} \quad g^{22} = \frac{1}{r^2} \quad g^{33} = -\frac{1}{a^2 r^2 \sin^2 \theta} \]

We'll also need \( \sqrt{-g} \) (the determinant) \( = e^{-3r_0} \frac{\sin \theta}{\sqrt{1 - kr^2}} \)

Taking the definition

\[ \Pi^{i \mu}_{\nu \rho} = -g^{ik} \left[ -\frac{\partial g_{\mu \rho}}{\partial x^k} + \frac{\partial g_{\mu k}}{\partial x^\rho} + \frac{\partial g_{k \rho}}{\partial x^\mu} \right] \]

We can verify that there are only 12 non-zero affine connections.

\[ \Pi^0_0 = \frac{1}{c^2} \quad \Pi^1_1 = \frac{a^2}{c(1-kr^2)} \quad \Pi^2_2 = \frac{a^2 r^2}{c} \quad \Pi^3_3 = \frac{a^2 r^2 \sin^2 \theta}{c} \]

\[ \Pi^{11} = \frac{kr^2}{1-kr^2} \quad \Pi^{12} = \Pi^{13} = \frac{1}{r} \]

\[ \Pi^{22} = r(1-kr^2) \quad \Pi^{23} = -r(1-kr^2) \sin^2 \theta \]

\[ \Pi^{33} = \sin \theta \cos \theta \quad \Pi^{23} = c \cot \theta \]

From these we can build \( R_{\mu \nu} \) and \( \mathbf{R} \). For \( R_{\mu \nu} \) a useful relation is:

\[ R_{\mu \nu} = \frac{\partial^2 \ln(E)}{\partial x^\mu \partial x^\nu} + \Gamma^m_{\mu \rho} \Gamma^n_{\nu \rho} - \Gamma^l_{\mu \rho} \Gamma^m_{\nu \lambda} \frac{\partial \ln(E)}{\partial x^l} \]

\( \Gamma^i_{\mu \nu} = g^{i \mu} R_{\nu \lambda} \)
After a lot of algebra, it turns out only 4 elements of $R^k_i$ are non-zero.

As expected, the off-diagonal terms are all zero, and the spatial ones are identical:

$$R^0_0 = \frac{3}{c^2} \frac{\ddot{a}}{a}$$

$$R^1_1 = R^2_2 = R^3_3 = \frac{1}{c^2} \left( \frac{\dddot{a}}{a} + 2 \frac{\ddot{a}^2}{a^2} + 2 \frac{k c^2}{a^2} \right)$$

Hence, the Ricci scalar is

$$R = \frac{6}{c^2} \left( \frac{\dddot{a}}{a} + \frac{\ddot{a}^2}{a^2} + \frac{k c^2}{a^2} \right)$$

So

$$G^0_0 = R^0_0 - \frac{1}{2} R = -\frac{3}{c^2} \left( \frac{\dddot{a}}{a^2} + \frac{k c^2}{a^2} \right)$$

$$G^1_1 = G^2_2 = G^3_3 = R^1_1 - \frac{1}{2} R = -\frac{1}{c^2} \left[ \frac{2 \dddot{a}}{a} + \frac{\ddot{a}^2}{a^2} + \frac{k c^2}{a^2} \right]$$

The $G^0_0 = \frac{8 \Pi \sigma}{c^4}$ equation now looks like

$$-3 \left( \frac{\dddot{a}}{a} + \frac{k c^2}{a^2} \right) = \frac{8 \Pi \sigma}{c^4}$$

The second equation (the $G^1_1$ equation) can be solved by subbing in

$$G^1_1 = \frac{8 \Pi \sigma}{c^4} \Rightarrow -\frac{1}{c^2} \left[ \frac{2 \dddot{a}}{a} + \frac{\ddot{a}}{a} + \frac{k c^2}{a^2} \right] = \frac{8 \Pi \sigma}{c^4}$$

$$-\frac{1}{c^2} \left[ \frac{2 \dddot{a}}{a} - \frac{8 \Pi \sigma}{c^2} \right] = \frac{8 \Pi \sigma}{c^4}$$

$$\Rightarrow \dddot{a} + \frac{8 \Pi \sigma}{a} = \frac{8 \Pi \sigma}{c^2} \frac{p + \frac{3 \Pi \sigma}{c^2}}{2}$$

or

$$\dddot{a} + \frac{8 \Pi \sigma}{a} = \frac{8 \Pi \sigma}{c^2} \frac{p + \frac{3 \Pi \sigma}{c^2}}{2}$$

$$\Rightarrow \frac{2 \dddot{a}}{a} = -\frac{8 \Pi \sigma}{c^2} p - \frac{8 \Pi \sigma}{c^2} p = -\frac{8 \Pi \sigma}{c^2} \left[ p + \frac{3 \Pi \sigma}{c^2} \right]$$
Lecture 8.2

These are two of the fundamental equations of Cosmology -
The Friedmann equation and the acceleration equation:

- A third is needed in general, an equation of state:
  \[ P(v) \]
  - for "dust" \( P = 0 \)
  - for relativistic particles (and photons) \( P = \frac{\rho}{3} \)
  - Cosmological constant \( P = -\Lambda \)

in general
\[ P = \omega \rho \]
\( \omega \) - equation of state parameter.

With these 3 equations, we can solve for the evolution of the Universe

\[
\left( \frac{\dot{a}}{a} \right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3} \rho
\]

\[
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left[ P + 3\rho \right]
\]

\[ P = \omega \rho \]

Example: Let's imagine an empty Universe \( g = P = 0 \)

\[
\left( \frac{\dot{a}}{a} \right)^2 = -\frac{kc^2}{a^2} \rightarrow \ddot{a} = -\frac{kc^2}{a^2} \text{ [also constant]}
\]

\[ \ddot{a} = 0 \]

\[ \rightarrow a \rightarrow \text{ constant expansion } a(t) \rightarrow T
\]

Before we go into that, let's talk about the RW metric:

A few observations:

First, there's an alternate form of the metric:

if you make the coordinate transformation

\[ x = \int \frac{dr}{\sqrt{1 - kr^2}} \text{ then } \Delta x = \frac{dr}{\sqrt{1 - kr^2}} \]
Lecture 8.3

The integral can be evaluated for different \( k \):

- \( k = 0 \):
  \[ X = \int \frac{d\tau}{\sqrt{1 - \tau^2}} = \tau \]

- \( k = -1 \):
  \[ X = \int \frac{d\tau}{\sqrt{1 + \tau^2}} = \sinh \tau \]

- \( k = 1 \):
  \[ X = \int \frac{d\tau}{\sqrt{1 - \tau^2}} = \sin \tau \]

\[ r = X \]
\[ r = \sinh x \]
\[ r = \sin x \]

\[ ds^2 = c^2 dt^2 - a^2(t) \left( \frac{dx^2}{\sinh^2 x} + \frac{dz^2}{\sin^2 z} + \frac{d\theta^2 + \sin^2 \theta d\phi^2}{\sin^2 \theta} \right) \quad (k = 0) \]

\[ ds^2 = c^2 dt^2 - a^2(t) \left[ d\tau^2 + \sinh^2 x \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right] \quad (k = -1) \]

\[ ds^2 = c^2 dt^2 - a^2(t) \left[ d\tau^2 + \frac{d\theta^2 + \sin^2 \theta d\phi^2}{\sin^2 \theta} \right] \quad (k = 1) \]

\[ S_{k}(\tau) = \begin{cases} x & (3/2 \quad k = 0) \\ \sinh x & (k = -1) \\ \sin x & (k = 1) \end{cases} \]

Let's think about the "surface area" as a spatial "hypersurface" (\( dt = 0 \))

\[ \int_{0}^{\frac{\pi}{2}} \int_{0}^{2\pi} \int_{0}^{a(t)} \frac{d\tau}{S_{k}(\tau)} \frac{d\theta}{\sin \theta} d\phi = \int_{0}^{\frac{\pi}{2}} \int_{0}^{2\pi} d\theta d\phi \frac{a(t)}{S_{k}(\tau)} \]

\[ A_{\text{area}} = 4\pi a(t) S_{k}(\tau) \]

These are a few results that are relevant to Cosmology that arise directly from \( a(t) \): the redshift and time dilation.
lecture 9-1

Consider light from some distant galaxy traveling to you: you are at \( r_0 \), \( \theta_0 \), \( \phi_0 \). For simplicity, the galaxy will be at \( r_1 \), \( \theta_0 \), \( \phi_0 \).

Light travels on null geodesics.

\[ ds^2 = 0 \]

but \[ ds^2 = c^2 dt^2 - a^2(t) dr^2 = 0 \]

\[ c \frac{dt}{a(t)} = \frac{dr}{\sqrt{1 - kr^2}} \]

We can separate the \( r \)-dependent and \( t \)-dependent parts:

\[ \int \frac{c \ dt}{a(t)} = \int \frac{dr}{\sqrt{1 - kr^2}} \]

What’s significant about this? First, let’s consider two “crests of a light wave,” emitted at two (nearly simultaneous) times \( t_e \) and \( t_e + \Delta t \). The null geodesics are:

\[ \int_{t_e}^{t_e + \Delta t} c \frac{dt}{a(t)} = \int_{r_e}^{r_e + \Delta r} \frac{dr}{\sqrt{1 - kr^2}} \]

and

\[ \int_{t_e}^{t_e + \Delta t} c \frac{dt}{a(t)} = \int_{r_e}^{r_e + \Delta r} \frac{dr}{\sqrt{1 - kr^2}} \]

are the same! (As long as the galaxy is at rest in comoving coordinates.)

so

\[ \int_{t_e}^{t_e + \Delta t} c \frac{dt}{a(t)} = \int_{t_0}^{t_0 + \Delta t} c \frac{dt}{a(t)} \]

most of this is the same. The only non-trivial part is
Lecture 9 p 2

\[ \int_{t_0}^{t} \frac{dt}{a(t)} = \int_{t_0}^{\tau t} \frac{dt}{a(t)} \]

but these are such small time stretches that \( a(t) \) is constant for then

\[ \frac{1}{a(t)} \int_{t_0}^{\tau t} dt = \frac{1}{a(t_0)} \int_{t_0}^{\tau t} dt \rightarrow \frac{1}{a(t_0)} \Delta \tau = \Delta \tau \]

So if you have events that are separated by \( \Delta \tau \) when they are emitted, they will be observed to be emitted at

\[ \Delta \tau = \Delta \tau \frac{a(t_0)}{a(t_0)} \]

this is also true for successive light ray... at the time between crests measures

\[ \lambda_0 = \Delta \tau = \frac{a(t_0)}{a(t_0)} \]

but \( \frac{\lambda_0}{\lambda} = \frac{1}{1+z} \) the redshift.

So \( \frac{\lambda_0}{\lambda} = 1 + z \), but from above:

\[ \frac{a(t_0)}{a(t_0)} = (1+z) \]

The factor \( H \tau \) measures the change in size of the universe.

That is what the cosmological redshift measures

One further consequence is that the energy carried by the photons is \( E = \frac{h}{\lambda} \).

E.g., \( a(t_0) \frac{1}{a(t_0)} (1+z) \). Let's consider

next what happens to an individual photon but to a collection.
Let's consider a distribution of particles with some volume \( d^3\mathbf{r} \) and with momenta in \( d^3\mathbf{p} \) (photons have \( \frac{1}{2} \mathbf{p} \times \mathbf{d}\mathbf{p} \), recall)

then we can write

\[
\frac{dN}{d^3\mathbf{p}} = \mathcal{F}(\mathbf{r}, \mathbf{p}, t) \, d^3\mathbf{r} \, d^3\mathbf{p}
\]

how does \( d^3\mathbf{r} \) transform with time?

\[
\mathcal{F}(\mathbf{r}, \mathbf{p}, t) \sim d^3\mathbf{r} \sim a^3(t)
\]

but \( d^3\mathbf{p} \sim V^3 \propto \frac{1}{V} \propto \frac{1}{a^3(t)} \)

So \( d^3\mathbf{r} \, d^3\mathbf{p} \) is constant. If the reaction conserves number (free streaming) \( dN = \text{constant} \)

\[
\mathcal{F}(\mathbf{r}, \mathbf{p}, t) = \text{constant in } t.
\]

We can turn this around:

\[
\int \frac{dN}{d^3\mathbf{p}} = \text{constant} \implies \frac{dN}{d^3\mathbf{r}} \sim \frac{d^3\mathbf{p}}{d^3\mathbf{r}}
\]

for photons streaming

\[
d^3\mathbf{r} \propto d^3\mathbf{r} \, c \, d\lambda
\]

\[
d^3\mathbf{p} \propto V^3 \, d\lambda \, d\Omega
\]

\[
\implies \frac{dN}{d\lambda \, d\Omega \, c \, d\lambda} = \text{constant}
\]

but \( E = N \, h \nu \)

\[
\frac{dE}{d\lambda} = dN \, h \nu \quad dN \propto \frac{dE}{d\lambda}
\]

\[
\implies \frac{dE}{d\lambda \, d\Omega \, d\lambda \, c \, d\lambda} = \text{constant}
\]

but \( \frac{dE}{d\lambda \, d\Omega \, d\lambda \, c \, d\lambda} \equiv I_\nu \) the specific intensity

\[
\implies I_\nu / V^3 = \text{constant}
\]
Now let's think back to our specific intensity

\[ I_{\nu} = \frac{J}{4\pi} \text{ energy/}\text{m}^2/\text{sec/steradian}/\text{Hz} \]

Let's consider an infinitesimal range in frequency \( d\nu \)

\[ I_{\nu} d\nu \rightarrow \frac{J}{4\pi} \text{ energy/}\text{m}^2/\text{sec/steradian} \]

There is a specific form of \( I_{\nu} \) we're interested in: the blackbody function.

\[ I_{\nu} = B_{\nu} = \frac{2\pi}{C^2} \frac{\nu^3 d\nu}{e^{\frac{h\nu}{kT}} - 1} \]

Let's consider an isotropic radiation field.

The number of photons crossing the surface of a sphere of volume \( dV \) is (with frequency \( \nu/d\nu \))

\[ \int I_{\nu} \cos \theta d\Omega d\nu = 4\pi I_{\nu} \cos \theta d\Omega d\nu = 8\pi h \frac{C^2}{C^2} \frac{\nu^3 d\nu}{e^{\frac{h\nu}{kT}} - 1} \]

The number of photons is found by dividing by \( h\nu = \epsilon \)

\[ \frac{8\pi}{C^2} \frac{\nu^2 d\nu}{e^{\frac{h\nu}{kT}} - 1} \]

But photons travel at \( v = c \), so the number density inside the sphere is just the number density crossing the sphere.

\[ \rho(\nu, d\nu) = \frac{8\pi}{C^3} \frac{\nu^2 d\nu}{e^{\frac{h\nu}{kT}} - 1} \]

so

\[ \rho(\nu, d\nu) dV = \frac{8\pi}{C^3} \frac{\nu^2 d\nu}{e^{\frac{h\nu}{kT}} - 1} \]

\[ dV \text{ is } N = \int \rho(\nu, d\nu) dV = \frac{8\pi}{C^3} \frac{\nu^2 d\nu}{e^{\frac{h\nu}{kT}} - 1} \]

Oke, what happens to these photons as the universe expands
\[ \text{Well} \]

\[ dV = \frac{dV_0}{a_j^3} = \frac{dV_0}{(1+z)^3} \]

\[ V = V_0 (1+z) \]

\[ dV = dV_0 (1+z) \]

and \( N = N_0 \) (conservation of number)

\[ N = \frac{8 \pi}{C^3} \frac{V^2 J dV dV_0}{e^{\frac{hV}{kT}} - 1} = N_0 = \frac{8 \pi}{C^3} \frac{V_0^2 J dV_0 dV_0}{e^{\frac{hV_0}{kT_0}} - 1} \]

all the factors of \( (1+z) \) cancel out except the one in the exponential

\[ \Rightarrow \frac{hV}{kT} = \frac{hV_0}{kT_0} = \frac{hV_0 (1+z)}{kT} \]

\[ \Rightarrow T = T_0 (1+z) \]. But, because \( V_0 \) is conserved by the expansion

A blackbody distribution stays a blackbody distribution.

(back to 10.2)
Lecture 10 p2

10 $I_\nu /\nu^3$ is a constant. Why is it important?

Consider the Planck distribution:

The specific intensity $I_\nu = B_\nu \equiv 2\pi \frac{\hbar \nu^3}{c^2} \left( \frac{1}{e^{\hbar \nu/kT} - 1} \right)$

but

$$\frac{B_{\nu} \text{emitted}}{\nu \text{emitted}^3} = \frac{2\pi \nu^3}{c^2} \left( \frac{1}{e^{\hbar \nu/kT} - 1} \right) = \frac{2\pi h \nu^3}{c^2} = \frac{2h \nu^3}{c^2 e^{\hbar \nu/kT} - 1}$$

$$\frac{h \nu kT}{c^2} = \frac{h \nu kT_0}{c^2} \rightarrow \frac{\nu}{T} = \frac{\nu}{T_0}$$

$$\text{but } \nu = \frac{\nu}{1+z} \Rightarrow c \nu e = \lambda_0 = \lambda(1+z)$$

$$T_0 = T_e \left( \frac{\nu}{c} \right) = T_e / (1+z) \rightarrow \text{A Planck distribution stays a Planck distribution, but with } T \to \frac{1}{1+z}$$

(THIS is why the CMB temperature is 2.7 K today).

Oka, now back to our galaxy.

Our equation for

The geodesic for light dives

$$\int dt / \alpha(t) = \int \frac{dr}{\sqrt{1 - kr^2}} = S_k(p)$$

Padmanabhan derives at this point

$$d_l (z) = a \frac{d_z}{dz} \frac{d_z}{dz}$$

$$da = a_0 \frac{d_1 / (1+z)}{dz}$$

since $dz / dt = dz / d_z \frac{d_1 / (1+z)}{1+z}$
Lecture 10.3

\[
\frac{dt}{dz} = - \frac{dH(z)}{H(z)} \frac{dz}{dz} = \int_0^z \frac{dt}{dt'} \frac{dH(z)}{H(z)} = \frac{1}{a_0} \int_0^z \frac{dH(z)}{H(z)} \, dz
\]

So \( dp(z) = \frac{a_0}{a} \frac{1}{z} \int_0^z dH(z) \, dz \).

\( dp(z) \) is an interesting quantity, but is not an observable → it is measured at constant \( t \). The ways distances are observed are observed:
1) Luminosity distance \( d_L \):

\[
L \text{ light spreads out} \quad F = \frac{L}{4 \pi a_0^2} \quad \text{as area} \quad \frac{dL}{dt} = \frac{dL}{dt} (1 + z)
\]

\[
F = \frac{dL}{dt} = \frac{dL}{dt} \frac{dt}{dt} (1 + z)^2 = \frac{dL}{dt} \frac{dt}{dt} (1 + z)^2
\]

\[
\rightarrow dc = c dt = dp (1 + z)
\]

However, in general, we observe \( F \) in some frequency range \( \bar{F} \):

\[
\int \bar{F} \, dp (1 + z)
\]

The second form is the angular diameter distance \( d_A \):

\[
\bar{F} = \frac{D}{dp} \quad (\text{for } D < dx)
\]

The physical size subtended is \( \Theta \) at \( t = t_0 \):

\[
\Theta = \frac{D}{dp} \quad (\frac{dx}{dp})_{x=0} = \frac{D (1 + z)}{dp}
\]

\[
\rightarrow \frac{dx}{dp} = \frac{dp}{dt} (1 + z) \rightarrow \frac{dx}{dp} = \frac{dc (1 + z)^2}{dp}
\]
Lecture 10.4

Two more very brief topics.

1) Horizons -
   Let's go back to the equation we derived for a geodesic:

   \[ C \int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_0^r \frac{dr}{\sqrt{1-kr^2}} = S_k(r_g) \]

   \[ \Rightarrow r_g = S_k \left( C \int_{t_e}^{t_0} \frac{dt}{a(t)} \right) \]

   Let's consider

   \[ k=1 \] then \( S_k \) has a maximum value.

   \[ \Rightarrow r_g \] will have a maximum value. \( dp \) is \( r_g \) has a maximum, too.

   In general, for any universe with a twin, there will be a \( dp_{\text{max}} \):

   \[ dp_{\text{max}}(t) = a(t) S_k \left( C \int_{t \text{ twin}}^{t_e} \frac{dt}{a(t)} \right) \]

   This is the particle horizon.

   \[ \text{Similarly, if } t=t_{\text{max}} \]

   there is a \( dp_{\text{max}}(t) = a(t) S_k \left( C \int_{t_{\text{max}}}^{t_e} \frac{dt}{a(t)} \right) \]

   where events \( dt \)

   \[ dp > dp_{\text{max}} \] will not be seen by us - this is the event horizon.
Lecture 16.5
Practical Considerations

In all the previous sections, we've dealt with terms in terms of integrals of $a(t)$. In the next section, we'll consider solutions of the Friedmann equation that give $a(t)$. Different cosmological models.

However, you can be more specific for small $z$ (or $\Delta t \approx t$ small).

Taylor expand $a(t)$ around $a(t_0)$

$$a(t) = a(t_0) + \frac{d}{dt}(t-t_0) + \frac{1}{2} \frac{d^2}{dt^2}(t-t_0)^2 + \cdots$$

$$a(t) = a(t_0) \left[ 1 + \frac{\dot{a}}{a}(t-t_0) + \frac{1}{2} (t-t_0)^2 \frac{\ddot{a}}{a} + \cdots \right]$$

where $\dot{a} = \frac{\dot{a}}{a} \mid_{t=t_0}$ (Hubble's constant) $q_0 \equiv \frac{\ddot{a}}{\dot{a}^2}$

This expansion is valid as long as $(t-t_0)H_0$ is small:

$$\int_{t_0}^{t} \frac{dt}{a(t)} \approx \frac{1}{H_0}$$

With this expansion, we can integrate

$$\int_{a(t_0)}^{a(t)} dt = c \left[ z - z_0 \right] + \cdots$$

but

$$\frac{a(t)}{a(t_0)} = \frac{1}{H_0^2}$$

so the equation above to give $\Delta \theta$.

After some algebra:

$$\int_{a(t_0)}^{a(t)} dt = c \left[ z - \frac{1}{2} (1+q_0) z^2 + O(z^3) \right]$$

This recovers for small $z$ our linear relation (recall $z = \text{the doppler effect}$ $z = \frac{v}{c}$: $v = Ho$).
10.11

Friedmann-ology:

Let's return to our Friedmann and acceleration equations:

**Friedmann Eqn:**

\[
H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \rho - \frac{kc^2}{a^2}
\]

\[
\frac{\dot{a}}{a} = -\frac{4\pi G}{3} \left[ \rho + \frac{3}{c^2} \dot{a} \right]
\]

\[
\rho = \omega \rho c^2
\]

We can rewrite the Friedmann equation in terms of the density parameter \( \Omega \).

If we define \( \Omega_c = \frac{3H^2}{8\pi G} \) to be the critical density, then

The Friedmann Equations can be written as

\[
H^2 \left[ 1 - \frac{\rho}{\Omega_c} \right] = -\frac{kc^2}{a^2}
\]

or

\[
\frac{kc^2}{a^2} = H^2 \left[ \Omega - 1 \right]
\]

where \( \Omega \) (sometimes written as \( \rho \)) is the density parameter.

Note that because this is true at all times \( \Omega = \frac{\rho}{\rho_c} = \frac{\rho}{\rho_c} \),

It's true today

\[
\frac{kc^2}{a_0^2} = H_0^2 \left[ \Omega_0 - 1 \right]
\]

This relation says that \( k \) and \( \Omega_0 \) determine each other.
\[ k = 1 \rightarrow \Omega_0 > 1 \\
\] 
\[ k = -1 \rightarrow \Omega_0 < 1 \\
\] 
\[ k = 0 \rightarrow \Omega_0 = 1 \]

This equation also gives you \( a_0 \)
\[ a_0 = \frac{c^2}{H_0^2} \left( -2 \ln(1) \right)^{-\frac{1}{2}} \quad (\text{is } k \neq 0) \]

To go further, we have to understand the function \( g(a) \) (and \( P(a) \)).

A customary way to do this is to consider the first law of thermodynamics:
\[ dQ = dE + PdV \]

If the universe is expanding homogeneously, there is no bulk flow of heat \( dQ = 0 \rightarrow \text{adiabatic} \)

So
\[ \frac{dE}{dt} + P \frac{dV}{dt} = 0 \]

\[ E \propto \rho c^2 V \quad \text{so} \quad \frac{dE}{dt} = \rho c^2 V \frac{dP}{dt} + P \frac{dV}{dt} \]

\[ \frac{dV}{dt} \rightarrow V a \frac{da}{dt} \quad \text{so} \quad \frac{dV}{dt} = 3a^2 \frac{d(a)}{dt} = 3a^2 \left( \frac{a}{a} \right) \Rightarrow \frac{dV}{dt} = 3 \frac{d(a)}{dt} \]

\[ \rightarrow \text{So for an adiabatic flow} \]
\[ \left[ \rho \frac{c^2}{a} + \frac{3a^2 P a^2}{a^2} + \frac{P a^2}{a} \right] V = 0 \]

\[ \rightarrow \dot{S} = \frac{3a}{a} \left[ \rho + P c^2 \right] \quad \text{This is known as the fluid equation} \]
The fluid equation is not completely independent of the acceleration & equations of state

Fluid eqn $\Rightarrow$ Friedmann eqn

But it’s nice that they are consistent

Let’s imagine we have an equation of state $P = \omega P c^2$, so $P/c^2 = \omega P$

The fluid equation then says

$$\dot{a} = -3 \frac{\dot{a}}{a} \left[ 1 + \omega \right] \dot{a}$$

$$\Rightarrow \frac{\dot{a}}{a} = -3 \left[ 1 + \omega \right] \frac{\dot{a}}{a} \quad \text{for constant } \omega$$

This has an extremely simple solution:

$$a = \left[ \frac{a}{a_0} \right]^{-3(1+w)}$$

For matter $\omega = \frac{\rho}{c^2} = 0$ (Baryons, cold dark matter)

$$a = a_0 \left[ \frac{a}{a_0} \right]^{-3} \quad \text{(just volume expansion)}$$

For radiation

$$P = \frac{\rho c^2}{3} = \frac{\epsilon}{3} \Rightarrow \omega = \frac{1}{3}$$

$$a = a_0 \left[ \frac{a}{a_0} \right]^{-4} \quad \text{(volume expansion + redshift)}$$
11.4

For a "dark energy" with $\Omega = 1$,

$$(\rho = -\rho c^2) \rightarrow$$

$$\Omega = \Omega_0 (a_0 / a) = \rho / \rho_c \rightarrow \text{a constant}.$$ 

What does this mean?

$$dQ = 0 \rightarrow dE + P dV = 0$$

Fluid equation

$$\dot{\rho} c^2 = 3\dot{a} a \left[ (\rho + P) / c^2 \right]$$

If it's a constant density, LHS is zero.

1st law of thermodynamics is satisfied (for $P = -\rho c^2$).

Example of a scalar field:

$$P c^2 \sim \frac{1}{2} \dot{\phi}^2 + V(\phi)$$

$$P \approx \frac{1}{2} \dot{\phi}^2 - V(\phi)$$

For $\dot{\phi} \ll V(\phi)$:

$$\dot{\rho} c^2 \approx -P$$

$\rightarrow \omega \rightarrow -1$

If we consider a universe made up of lots of components:

$$\Omega (a) = \Omega_R + \Omega_{NR} + \Omega_{\phi}$$

$$\approx \Omega_{R}(a_0)^{-4} + \Omega_{NR}(a_0^{-3}) + \Omega_{\phi}$$

$$= \Omega_{R} \left[ \frac{\rho_{R}(a)}{\rho_c (a_0)^4} + \rho_{NR}(a_0^{-3}) + \rho_{\phi}(a_0)^{-1} \right]$$

$$= \rho_c \left[ \frac{\rho_{R}(a)}{\rho_c (a_0)^4} + \rho_{NR}(a_0^{-3}) + \rho_{\phi}(a_0)^{-1} \right] = \rho_c \left[ \Omega_{R}(a_0)^{-4} + \Omega_{NR}(a_0^{-3}) + \Omega_{\phi} \right]$$
\(11.5\)

Plugging this back into the Friedmann equation

\[
H^2 + \frac{kc^2}{a^2} = H_0^2 \left[ \Omega_{\text{Rho}} \left( \frac{a_0}{a} \right)^4 + \Omega_{\text{NRho}} \left( \frac{a_0}{a} \right)^3 + \Omega_{\text{Vac}} \right]
\]

We can go further and recall that

\[
k^2 c^2 = H_0^2 \left( \frac{a^2}{a_0^2} \right) = \frac{\dot{a}}{a}
\]

\[
\Omega_{\text{Rho}} = \frac{\frac{k c^2}{a^2}}{\left( \frac{a_0}{a} \right)^2} \left( \frac{H_0^2}{\Omega_{\text{Rho}}^2} - 1 \right)
\]

\[
H_0^2 = H_0^2 \left[ \Omega_{\text{Rho}} \left( \frac{a_0}{a} \right)^4 + \Omega_{b} \left( \frac{a_0}{a} \right)^3 + \Omega_{\text{Vac}} + \left( 1 - \Omega_{\text{Rho}} \right) \left( \frac{a_0}{a} \right)^2 \right]
\]

Eqn. \(H(a)\)

Similarly, you can turn this into an eqn. for \(\Omega(a)\)

\[
\Omega(a) = \frac{\dot{a}}{a} - 1
\]

\[
+ \frac{1 - \Omega_{\text{Rho}} + \Omega_{b} + \Omega_{\text{Vac}} - \Omega_{\text{Vac}}}{1 - \Omega_{\text{Rho}} + \Omega_{b} + \Omega_{\text{Vac}} - \Omega_{\text{Vac}}}
\]

as \(a \to 0\) the denominator diverges; so \(\Omega(a) \to 1\) for \(a \to 0\)

(BC of radiation)

This implies that \(\frac{k c^2}{a^2} \to 0\) as \(a \to 0\) (interesting point)

We're now ready to tackle solutions to the Friedmann eqn

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{\ddot{a}}{a} = H_0^2 \sqrt{F(a)}
\]

\[
\frac{da}{a} = H_0 \sqrt{F(a)} dt \Rightarrow \int_0^\tau \frac{da}{a} = \int_0^\tau \frac{dt}{H_0 \sqrt{F(a)}}
\]
11.6

\[ t = \int \frac{da}{a \sqrt{F(a)}} \]

In general, this has to be evaluated numerically. Let's take some cases:

1) \( S_r < \Sigma_{n,0} = \Sigma_{r,0} = 0 \)

\[ t = \frac{1}{H_0} \int \frac{da}{a \sqrt{\Omega_{r,0} a_0^3}} = \frac{1}{H_0 \sqrt{\Omega_{r,0} a_0}} \int da \]

\[ t(\infty) = \frac{1}{H_0 \sqrt{\Omega_{r,0} a_0}} \]

\[ \alpha \rightarrow \frac{\alpha}{a_0} \quad H_0 \sqrt{\Omega_{r,0}} \quad \text{trivial} \]

\[ \alpha \rightarrow a_0 \quad \text{hot} \quad \text{but} \quad 1 = \Sigma_{r,0} + \Sigma_{n,0} + \Sigma_{\Lambda,0} + \Sigma_{k,0} \]

Radiation.

This is important because

\[ \lim_{a \rightarrow 0} \frac{\Theta}{a} \rightarrow \frac{\Theta}{a} \quad \text{(since it has the highest density dependence on } a) \]

Therefore, the early universe is radiation dominated.

\[ \int_{t_0}^{t} dt = \frac{1}{H_0} \int_{0}^{a} \frac{da}{a \sqrt{\Omega_{r,0} a_0^3}} \]

\[ = \frac{1}{H_0} \int_{0}^{a} \frac{da}{\sqrt{\Omega_{r,0} a_0^2} \left( \frac{1}{1+z} \right)^2} \]

\[ = \frac{a^2}{2 H_0 a_0^2 \sqrt{\Omega_{r,0}}} \rightarrow \frac{(\frac{\alpha}{a_0})^2}{2 H_0 t \sqrt{\Omega_{r,0}}} \]

or

\[ (1+z) \alpha \times t^{\frac{1}{2}} \]
12.1

Note that there is an easy mapping between redshift and time (and temperature and time), since $T = T_0 (1+z)$.

Two more simple cases

$\frac{\text{NR matter}}{t} = \frac{1}{H_0} \int \frac{da}{a \sqrt{\Omega_{\text{m}} a^3 + \Omega_{\text{r}} a^3}}$

$= \frac{1}{H_0} \int \frac{a^{1/2} da}{\sqrt{\Omega_{\text{m}} a^3 + \Omega_{\text{r}} a^3}} = \frac{2}{3 H_0 \sqrt{\Omega_{\text{m}}}}$

$t = \frac{2}{3 H_0 \sqrt{\Omega_{\text{m}}}} \left( \frac{a}{a_0} \right)^{3/2}$, $a \propto t^{2/3}$

$t \propto (1+z)^{-3/2}$

In general, if $\omega \neq -1$

$a(t) \propto t^{2/3 (\Omega_0)}$

$\omega = -1$ is a special case:

$t = \frac{1}{H_0} \int \frac{da}{a \sqrt{\Omega_{\text{m}} a^3 + \Omega_{\text{r}} a^3}}$

$= \frac{1}{H_0} \left[ \ln \left( \frac{a}{a_{\text{min}}} \right) \right]_{\Omega_{\text{m}} = \Omega_{\text{r}}}$

If $a_{\text{min}} < 1$, then $t$ diverges.

$A \propto e^{H_0 t}$

$\rightarrow$ exponential growth.

These limiting solutions are valid for when one term in the "density" dominates over the others.
12.2

In general, you have different "eras"

\[ a \sim e^{\frac{t}{\chi}} \text{ radiation} e^{\frac{t}{\Omega}} \text{ matter} \]

\[ e^t \sim \text{ dark energy} \]

Note because any of the \( \Omega_{K,0} \) or \( \Omega_{\Lambda,0} \) could be zero you don't have to have these eras.

Furthermore even if they exist it's possible to "miss eras". For example in our universe \( \Omega_{K,0} = 0 \) and \( \Omega_{\Lambda,0} \approx 0.7 \) That means the universe transitioned directly from matter to DE -- it skipped the "curvature era".

The transitions are controlled by the values of the Ks. However, one point remains fixed: The Universe started radiation dominated (this will be important for Chapter 4).

In our specific Universe \( \Omega_{\Lambda,0} \) is high enough that we didn't have a "matter era", so there was a transition between radiation and matter, and at a time of matter-radiation equality, when

\[ P_r = P_{\Lambda,0} \text{ occurred} \]

\[ P_r(a_0) = P_{\Lambda,0} \left( \frac{a_0}{a} \right)^3 \longrightarrow \frac{1}{(1+z)^3} \frac{a_0}{a} = \frac{P_{\Lambda,0}}{P_{\Lambda,0}} \]

\( P_{\Lambda,0} \) can be determined because the majority of the energy density is in the CMB:
For a space filled with a blackbody
\[ P_r = \left( \frac{\pi^2}{15} \right) K B^4 T^4, \quad P_c = \frac{3h c^2}{8\pi^6} \]
\[ \Rightarrow \quad R_{R,0} = 2.56 \times 10^{-2} {h}^{-2} \]
\[ \Rightarrow (1 + \zeta) = 3.9 \times 10^{-4} R_{NR,0} {h}^2 \]
\[ T_{eq} = T_0 (1 + \zeta) \approx 1.06 \times 10^5 R_{NR,0} {h}^2 \]
\[ K = 9.24 (R_{NR,0} {h}) \]

2-component universes

It turns out that there is an analytic solution to \( t(a) \) is only matter and radiation exist:

\[ H_{eq} t = \frac{2}{3} \left[ \left( \frac{a}{a_{eq}} - \frac{1}{3} \right) \left( \frac{a}{a_{eq}} + 1 \right)^{1/2} + 2 \right] \]

How did this get here:

\[ \left( \frac{a}{a_c} \right)^2 = H_0^2 \left[ \Omega_R (\frac{a_c}{a})^4 + \Omega_N (\frac{a_c}{a})^3 \right] \]

\[ \int dt = H_0^2 \int \frac{da}{a} \sqrt{R_{LR}(a) + R_{NR}(a)} \]

but \( a_0 \left( \frac{R_{NR}}{\rho_{eq}} \right) \approx a_{eq} \)

So this is:

\[ \mathcal{H} \left( \frac{a}{a_c} \right) = \frac{1}{H_0 a_0^2 \sqrt{R_{LR}}} \int \frac{a da}{\sqrt{1 + \frac{a}{a_{eq}}} x} \Rightarrow \text{for } \int x dx \frac{1}{1 + bx} \]

This integral can be looked up; its answer is

\[ -a \left( 2 \frac{a}{a_c} \right) a_{eq}^2 \left( 1 + \frac{a}{a_c} \right)^{1/2} + C \]

Requiring \( t = 0 \) when \( a = a_0 \), yields our constant:

\[ C = \frac{1}{3} a_{eq}^2 \]
12.4

So (factor out $a_e^2$)

$$
\epsilon = \frac{1}{H_0 \alpha^2 \sqrt{\Omega_R}} \left[ 2 \left( \frac{a}{a_{eq}} - 2 \right) \left( 1 + \frac{a}{a_{eq}} \right)^{\frac{1}{2}} + \frac{4}{3} \right]
$$

$$
= \frac{1}{H_0 \sqrt{\Omega_R}} \left( \frac{a_e^2}{\alpha^2} \right) \frac{2}{3} \left[ \left( \frac{a}{a_{eq}} - 2 \right) \left( 1 + \frac{a}{a_{eq}} \right)^{\frac{1}{2}} + 2 \right]
$$

Padmanabhan writes this in terms of $\text{Heq}^2$.

Using the Friedmann equation

$$
H^2 = \frac{8\pi G \rho}{3}
$$
at equality $\rho = \rho_R + \rho_{\text{neq}} = 2\rho_R = 2\rho_{\text{Ro}} (1 + \text{Heq})^4$

$$
\text{but } \rho_{\text{eq}} = 3H_0^2 \rightarrow \frac{8\pi G}{3} = \frac{H_0^2}{\rho_{\text{eq}}}
$$

So $\Omega_{\text{eq}}^2 = H_0^2 2\rho_{\text{Ro}} (1 + \text{Heq})^4$

$$
\rightarrow \text{Heq} = H_0 \sqrt{\rho_{\text{Ro}} (1 + \text{Heq})^2}
$$

$$
\rightarrow \epsilon = \frac{\sqrt{2}}{3 \text{Heq}} \left[ \left( \frac{a}{a_{eq}} - 2 \right) \left( 1 + \frac{a}{a_{eq}} \right)^{\frac{1}{2}} + 2 \right]
$$

So $t_{\text{eq}} = \frac{3 \sqrt{2}}{3 \text{Heq}} \left[ 2 - \sqrt{2} \right] \approx \frac{5\sqrt{2}}{\text{Heq}}$

For $k < 0$

$$
\text{Heq} = H_0 \sqrt{2\rho_{\text{Ro}} (1 + \text{Heq})^2}
$$

$$
\text{t}_{\text{eq}} = \frac{5\sqrt{2}}{\sqrt{2} H_0 \sqrt{2\rho_{\text{Ro}}}} (1 + \text{Heq})^{-\frac{3}{2}}
$$

but $(1 + \text{Heq}) \approx 3.9 \times 10^4 \left( \frac{\rho_{\text{neq}}^2}{H_0^2} \right)^{\frac{1}{2}}$

$$
H_0 \approx 3.1 \times 10^7 \text{ m/s}
$$

$$
\text{so } t_{\text{eq}} \approx 1.57 \times 10^{10} \left( \frac{\rho_{\text{neq}}^2}{H_0^2} \right)^{\frac{1}{2}} \text{ seconds} \lesssim 10^4 \text{ years}
$$