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PH 207 - ADVANCED QUANTUM MECHANICS

Prof. Guralnik $\rightarrow$ No exams
All homework, will be taken seriously

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Rotations in N dimensions
$(x_1, x_2, x_3, x_4, x_5, \_\_, x_N) \rightarrow (x)^T$
are described by $N \times N$ matrix R
$\Rightarrow (x') = R(x)$
with the constraint $\boxed{R^T R = I} \Rightarrow \det(R^T R) = \det I = 1$

$$(\det R^T)(\det R) = (\det R)^2 = 1$$

$\Rightarrow \det R = \pm 1$

$$R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det R = -1$$

$$'X = R X = \begin{pmatrix} x_1 \\ x_2 \\ \end{pmatrix} = \Big(-1$$

$('X) = R^1(X)$
$(''X) = R^2(X') = R^2R'(X)$

does R^2R^1 = Rotation Matrix?

$(R^2R')^T (R^2R') = R'^T R^{2T} R^2 R' = R'^T I R' = R'^T R' = I$ ✓

$\Rightarrow$ it is a rotation matrix

$\Rightarrow$ Rotation matrices form a group.

def: Group requires $R_1 \cdot R_2 = R_3 \Rightarrow$ closure
Identity $IR = RI = R$
Associativity $(R_1 R_2)R_3 = R_1(R_2 R_3)$
Inverse for each $R_1^T R_1 = I = R_1 R_1^T$. $\Rightarrow R_1^{-1} = R_1^T$

Problem I Prove that the inverse of a rotation is a rotation

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PH 207 → [illegible] Physics

$$S = \int_{t_1}^{t_2} dt \left[\vec{p}\dot{\vec{q}} - H(\vec{p}, \vec{q}) \right]$$

$$\delta S = \int \delta\vec{p} \left[\dot{\vec{q}} - \frac{\partial H}{\partial \vec{p}} \right] + \delta\vec{q} \left[-\dot{p} - \frac{\partial H}{\partial \vec{q}} \right]$$

$$\delta S = 0 \,, \quad \boxed{\dot{q} - \frac{\partial H}{\partial p} = 0, \quad \dot{p} + \frac{\partial H}{\partial q} = 0}$$

$$(X) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \,, \quad \begin{pmatrix} 'x_1 \\ 'x_2 \\ 'x_3 \end{pmatrix} = '(X) \rightarrow \text{rotated axis}$$

$$^T X X = x_1^2 + x_2^2 + x_3^2 = {}^T X' X' = 'x_1^2 + 'x_2^2 + 'x_3^2$$

$$'(X) = R(X) \quad \text{or} \quad 'X = RX \quad \text{(change notation)}$$

R ↳ 3x3 Matrix

$$^T X X = {}'X^T {}'X = X R^T R X$$

$$\Rightarrow R^T R = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem I: Show that If $R^T R = I$ then $R R^T = I$

$$R^T R = 1$$

$$\Rightarrow \det R^T R = 1 \,, \quad (\det R^T)(\det R) = 1$$

$$\Rightarrow (\det R)^2 = 1 \,, \quad \boxed{\det R = \pm 1} \checkmark$$

$$R\left(m\frac{d^2\vec{x}}{dt^2}\right)=0 \quad , \quad m\frac{R\,d^2\vec{x}}{dt^2}=0 \quad , \quad m\frac{d^2 R\vec{x}}{dt^2}=0$$

$$m\frac{d^2\vec{x}'}{dt^2}=0$$

Note: for $F=0$, define $\underbrace{\frac{1}{a}\vec{X}\equiv\vec{X}'}$ $\longrightarrow$ $m\frac{d^2\vec{x}'}{dt^2}=0$

↳ Scale transformation

$$S\overset{D}{=}\int_{t_1}^{t_2}dt\left[\vec{p}\cdot\frac{d\vec{q}}{dt}-H(\vec{p},\vec{q})\right]$$

If $H=\frac{p^2}{2m}$, Then $\Rightarrow$ $\frac{d^2\vec{q}}{dt^2}=0$

$$S=\int_{t_1}^{t_2}\left[\vec{p}\cdot\frac{d\vec{q}}{dt}-\frac{\vec{p}\cdot\vec{p}}{2m}\right]dt$$

Note: Under rotations where $\vec{p}'=R\vec{p}$, $\vec{q}'=R\vec{q}$

S is unchanged.

if

$H=P_x$ $\to$ not invariant under rotations

$$H(\vec{p},\vec{q})=H(\vec{p}\cdot\vec{p},\vec{p}\cdot\vec{q},\vec{q}\cdot\vec{q})$$

Rotations form a group:

A group is a set of elements $(a, b, c, d \ldots)$ with an operation $\cdot$

(i) $c \cdot e$ = an element in the set

(ii) has unit element I : $I \cdot b = b$
$b \cdot I = b$

(iii) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

(iv) must have inverse i.e., f, there is an f^{-1}
: $f^{-1} f = f f^{-1} = I$, f^{-1} is a member of the set.

Problem 2: Show the rotations ~~[illegible]~~ in any finite dimension form a group.

These are defined by the length preserving real linear transformations. i.e. $\vec{X}' = R\vec{X}$, $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

length preserving means $R^T R = I$

This group is called $\underline{O(n)}$ in n dimensions.

Special case: - no reflections

Problem: Show that if only $\det R = +1$ allowed, you still have a group called $SO(n)$ (S for special)

Lie groups are groups that can be characterized by numerical parameters and where the elements of the group are continuous functions of these parameters.

2-D

$$\left.\begin{aligned} x' &= \cos\theta\, x + \sin\theta\, y \\ y' &= y\cos\theta - x\sin\theta \end{aligned}\right\} \quad SO(2)$$

$$\bar{X}' = R\vec{X}, \quad R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\text{as } \theta \to 0, \quad R \Rightarrow \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix}$$

$$R = I + \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \left[I + \begin{pmatrix} 0 & d\theta \\ -d\theta & 0 \end{pmatrix} \right] \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{aligned} x' &= x + d\theta\, y \\ y' &= y - d\theta\, x \end{aligned}$$

, $d\theta$ for small θ

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + i\, d\theta \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\vec{X}' = \vec{X} + i\, d\theta\, \sigma_2 \vec{X}$$

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in general

$$(R)_{ij} = \underbrace{(1)_{ij}}_{\delta_{ij}} + \varepsilon_{ij} \quad \text{for a small rotation}$$

ε_{ij} is real, $R^T R = I$,

$$\boxed{(R^T{}_{il})(R)_{lm} = \delta_{im}}$$

$$(\delta_{li} + \varepsilon_{li})(\delta_{lm} + \varepsilon_{lm}) = \delta_{li}\delta_{lm} + \delta_{li}\varepsilon_{lm} + \delta_{lm}\varepsilon_{li} + \varepsilon_{lm}\varepsilon_{li}$$

$\Rightarrow$ assume that $\varepsilon_{lm}\varepsilon_{np} \ll \varepsilon_{qs}$

$$= \delta_{im} + \varepsilon_{im} + \varepsilon_{mi} = \delta_{im}$$

$$\Rightarrow \boxed{\varepsilon_{im} = -\varepsilon_{mi}} \longrightarrow \begin{pmatrix} 0 & \varepsilon & & \\ -\varepsilon & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \quad \text{antisymmetric}$$

How many independent elements in an antisymmetric $N \times N$ matrix?

$$\frac{N^2 - N}{2} = \frac{N(N-1)}{2}$$

<u>Note</u>, if $N=2 \Rightarrow$ # of independent elements 1

$N = 3 \Rightarrow$ # of " " 3

$N = 4 \Rightarrow$ # of " parameters 6

~~The~~ (infinetismall) Two dimensional rotation

$'x = x + \varepsilon_{12} y = x + d\theta y$
$'y = \varepsilon_{21} x + y = -d\theta x + y$

$$\begin{pmatrix} 'x \\ 'y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + i d\theta \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$(\vec{'X}) = (X) + i d\theta \, \sigma_2 (X)$$

$$\overset{(?)}{\Longrightarrow} \frac{d(X)}{d\theta} = i \sigma_x (X) \longrightarrow X = e^{i\sigma_2 \theta} (X^0)$$

Problem 4: Show that $(X) = e^{i\sigma_2 \theta} (X^0)$ describes an arbitrary rotation in 2 dimensions.

$R_\theta = e^{i\sigma_2 \theta}$
$R_{\theta_1} R_{\theta_2} = R_{\theta_2} R_{\theta_1}$

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n dimensional rotation

$R_{ij} = (1)_{ij} + \varepsilon_{ij}$

$R_{ij} = (1)_{ij} + \varepsilon_{\mu\nu} \left(\frac{J_{\mu\nu}}{2}\right)_{ij}$

$\Rightarrow \varepsilon_{\mu\nu} \left(\frac{J_{\mu\nu}}{2}\right)_{ij} = \varepsilon_{ij}$

$$(J_{\mu\nu})_{ij} = \left[\delta_{\mu i}\,\delta_{\nu j} - \delta_{\mu j}\,\delta_{\nu i}\right] \quad \#$$

$$\boxed{R = I + \frac{\varepsilon_{\mu\nu}}{2} J_{\mu\nu}}$$

not for summation

do 3-dimensions, as in all dimensions $J^{ii} = 0$

in n dimensions, there are $\frac{n^2-n}{2}$ independent $J^{\mu\nu}$

$$\to \quad J_{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$J_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

define (ε_i) 3 matrices

$$(\varepsilon_i)_{jk} = \varepsilon_{ijk}$$

$$J_{12} \equiv \varepsilon_3, \quad J_{13} = -\varepsilon_2, \quad J_{23} = \varepsilon_1$$

$$\delta w_1 = \varepsilon_{23}$$
$$\delta w_2 = -\varepsilon_{13}$$
$$\delta w_3 = \varepsilon_{12}$$
$$J_e \equiv \frac{\varepsilon_e}{i}$$

In this notation,

an arbitrary small 3 dim rotation is given by

$$\boxed{R = 1 + i\delta\vec{w}\cdot\vec{J}}$$

$$'X = \left(1 + i\,\delta\vec{w}_1\cdot\vec{J}\right) X$$

$$''X = \left(1 + i\delta\vec{w}_2\cdot\vec{J}\right) X'$$

$$\Rightarrow \quad \left(1 + i\delta\vec{w}_1\cdot\vec{J}\right)\left(1 + \delta\vec{w}_2\cdot\vec{J}\right) = \left(1 + i\,\delta\vec{w}_3\cdot\vec{J}\right)$$

since rotations form a group

$$(''X) = R_2 R_1 (X)$$

$$R_2R_1 - R_1R_2 = i(\delta W_2)_i\,(i\delta w_1)_j\,[J_iJ_j - J_jJ_i]$$

$$\Downarrow \qquad \Downarrow$$

$$\left(1 + i\delta\vec{w}_3\cdot\vec{J}\right) - \left(1 + i\delta\vec{w}_4\cdot\vec{J}\right) = i\left(\delta\vec{w}_3 - \delta\vec{w}_4\right)\cdot\vec{J}$$

$$\Rightarrow \quad [J_iJ_j - J_jJ_i] = i\,N_{ije}J_e$$

Problem 5: Prove that $\underbrace{[J_i, J_j] = i\,\epsilon_{ijk}\,J_k}$

structure of lie group

adjoint representation
of 3-d lie group

$\cancel{\text{XXX}}$ $\quad l^2 = X^2 = {'X}^2$

Lorentz Group) $(\vec{x}, \vec{y}, \vec{z}, ct)$

$$\parallel$$

$$(x^1, x^2, x^3, x^0)$$

$$(X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 = l^2$$

Lorentz transformation is a linear transformation l

$$(l)_{\mu\nu} \equiv l^{\mu}_{\cdot\nu}$$

$${'X}^{\mu} = l^{\mu}_{\cdot\nu} X^{\nu},$$

$${'X}^{\mu} g_{\mu\nu} {'X}^{\nu} = X^{\mu} g_{\mu\nu} X^{\nu}$$

$$\Rightarrow \left[{'X}^{0\,2} - ({'X}^1)^2 - ({'X}^2)^2 - ({'X}^3)^2 = (X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 \right]$$

$$g^{\mu\nu} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}_{(g)_{\mu\nu} = g_{\mu\nu} = g^{\mu\nu}} = g_{\mu\nu}$$

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$$S = \int dt \left(\vec{p}\cdot\dot{\vec{q}} - \frac{\vec{p}^{\,2}}{2R} - V(p^2, q^2, pq) \right)$$

$$(R)_{ik} = \delta_{ik} + \varepsilon_{ik} \quad , \quad \varepsilon_{ik} = -\varepsilon_{ki}$$
$$\equiv (1 + i\delta\vec{w}\cdot\vec{J})_{ik}$$

$$(J_i)_{jk} = \varepsilon_{ijk}$$

$$[J_i, J_j] = i\varepsilon_{ijk} J_k$$

$$\sum_{i=1}^{3} J_i J_i = 1.2 \begin{pmatrix} 1 & & \\ & 1 & \\ 0 & & 1 \end{pmatrix} = (j)(j+1) \quad , \quad j = 1$$

$[J^2, J_i] = 0 \Rightarrow$ diagonalize J^2, J_z at the same time.

$\Rightarrow$ Problem: Prove that the eigenvalues of $J_z = (1, 0, -1)$

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

$J_1 + iJ_2 \Rightarrow$ raising operator
$J_1 - iJ_2 \Rightarrow$ lowering operator

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2} \right] = i\epsilon_{ijk} \frac{\sigma_k}{2}$$ → 2x2 Pauli matrices

$$\sum_{i=1}^{3} \left(\frac{\sigma_i}{2} \right)^2 = \frac{1}{2} \cdot \frac{3}{2} I = \frac{3}{4} I \quad , \quad j = \frac{1}{2}$$

spin $= \frac{1}{2}$

$g^{\mu\nu} = (g)_{\mu\nu} \Rightarrow g^{00} = 1,\ g^{11} = g^{22} = g^{33} = -1$

$X_0 = X^0,\ X_1 = -X^1,\ X_2 = -X^2,\ X_3 = -X^3$

$X^\mu X_\mu = X^\mu g_{\mu\nu} X^\nu = (X)^T (g)(X) = S^2$

$(X)_\mu = X^\mu \quad , \quad (X) = \begin{pmatrix} X^0 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix}$

$S^2 = (X)^T (g)(X) = ('X)^T (g)('X)$

Now $'X^\mu = l^\mu_{\cdot\nu} X^\nu \Rightarrow ('X) = (l)(X)$

$\Rightarrow (X)^T (g)(X) = (X)^T (l)^T (g)(l)(X) = S^2$

<u>Note:</u> Why the dot? $l^\mu_{\cdot\nu} \Rightarrow$ because $g_{\alpha\beta} l^\beta_{\cdot\nu} = l_{\alpha\nu}$
We will show that in general $l_{\alpha\nu} \neq l_{\nu\alpha}$

$\Rightarrow \quad (l)^T (g)(l) = (g)$

$L^T g L = g$

$\det(L^T g L) = \det(L^T)\det(g)\det(L) = \det(g)$

$\Rightarrow \boxed{\det(L^T)\det(L) = 1} \qquad \boxed{\det(L) = \pm 1}$

proper transformation $\Rightarrow \det L = 1$ { do not allow time reflection and reflection of one or three coords.

improper $\Rightarrow \det L = -1$

$'X^{\mu} = \ell^{\mu}_{\cdot\nu} X^{\nu}$

Tensor of rank 2:

example $X^{\mu} Y^{\nu} = M^{\mu\nu}$

$$'X^{\mu}\,'Y^{\nu} = \ell^{\mu}_{\cdot\alpha} X^{\alpha} \ell^{\nu}_{\cdot\beta} X^{\beta}$$

$$\boxed{'M^{\mu\nu} = \ell^{\mu}_{\cdot\alpha} \ell^{\nu}_{\cdot\beta} M^{\alpha\beta}}$$

in general, if $\boxed{'T^{\mu\nu} = \ell^{\mu}_{\cdot\alpha} \ell^{\mu}_{\cdot\beta} = T^{\alpha\beta}}$, we say

$T^{\mu\nu}$ is a tensor of rank 2 (under Lorentz Transformations),

$'T = (\ell \times \ell)\, T \Rightarrow$ matrix notation for a tensor

Problem: Show the set of all LT form a group and the group is called "Lorentz Group".

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$$\underbrace{l^T g\, l = g}_{\downarrow} \quad , \quad \det l = \pm 1$$

$$g^{\mu\nu} = l^{\mu}_{\cdot\alpha}\, l^{\nu}_{\cdot\beta}\, g^{\alpha\beta}$$

Note: Arbitrary tensor of ~~xxxx~~ rank 2

$$-T^{\mu\nu} = l^{\mu}_{\cdot\alpha}\, l^{\nu}_{\cdot\beta}\, T^{\alpha\beta}$$

Note: Under spatial rotations $(\mathbb{I})_{\alpha\beta} = \delta_{\alpha\beta}$

$$R^T R = I \quad \Rightarrow \quad \delta_{\alpha\beta} = r_{\alpha\lambda}\, r_{\beta\gamma}\, \delta_{\lambda\gamma}$$
$$R^T I R = 1$$

Problem: Study ϵ_{ijk} (3 dim completely antisymmetric tensor) and show that this is an invariant tensor except for reflections.

$$\epsilon_{ijk} = r_{i\alpha}\, r_{j\beta}\, r_{k\lambda}\, \epsilon_{\alpha\beta\lambda}$$

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pseudovector!

$\epsilon_{ijk} \to$ invariant pseudo tensor of rank 3

Lorentz Transformations form a group

$L_1 L_2 = L_3 \to$ Lorentz group related to $SL(2, C)$

for rot. in 3-D

$$r_{ij} = \delta_{ij} + \varepsilon_{ij}$$

Lorentz

$$l_{\mu\nu} \equiv g_{\mu\alpha}\, l^{\alpha}_{\ \cdot\nu}$$

$$l'_{\mu\nu} = g_{\mu\nu} + \varepsilon_{\mu\nu} \quad \Longrightarrow \left(\text{~~neglect~~ valid only for proper L. G.} \right)$$

Note: could also write this as $l^{\mu}_{\ \cdot\nu} = g^{\mu}_{\ \cdot\nu} + \varepsilon^{\mu}_{\ \cdot\nu}$

$$\Downarrow$$

$$\delta^{\mu}_{\nu} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

Problem: Prove that for Lorentz Group

$$\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu} \quad \left(l'_{\mu\nu} = g_{\mu\nu} + \varepsilon_{\mu\nu} \right)$$

(ε) has $\frac{16-4}{2} = 6$ independent elements

3 of 6 parameters are rotation angles
3 of 6 parameters are called boost parameters

rotations in 3-D

$$R = 1 + \frac{i}{2} \sum \varepsilon_{ij} L_{ij} = 1 + i\, \delta\vec{w} \cdot \vec{J}$$

Lorentz

$$L = 1 + \frac{i}{2}\, \varepsilon_{\mu\nu} J^{\mu\nu} \quad \Longrightarrow \quad \left(J^{\mu\nu}\right)_{\alpha\beta} = i\left(\delta^{\mu}_{\ \cdot\alpha}\, \delta^{\nu}_{\ \cdot\beta} - \delta^{\mu}_{\ \cdot\beta}\, \delta^{\nu}_{\ \cdot\alpha} \right)$$

Note $\boxed{\delta^{\mu}{}_{\cdot\alpha} \equiv g^{\mu\beta}\delta_{\beta\alpha}}$

Note) A_{ij} is assumed antisymmetric

$$A_{ij} = -A_{ji}$$

S_{ij} is assumed symmetric.

Then $\sum_{ij} A_{ij} S_{ij} = 0$

Proof; $\displaystyle\sum_{ij} A_{ij}\,\frac{[S_{ij}+S_{ji}]}{2} = \sum_{ij}\left(\frac{A_{ij}S_{ij}}{2} + \frac{A_{ij}S_{ji}}{2}\right)$

$$= \sum_{ij}\left(\frac{A_{ij}S_{ij}}{2} - \frac{A_{ji}S_{ji}}{2}\right) = \sum_{ij}\left(\frac{A_{ij}S_{ij}}{2} - \frac{A_{ij}S_{ij}}{2}\right) = 0$$

$\Rightarrow$ $J^{\mu\nu}$ must be antisymmetric (symmetric part $\Rightarrow 0$)

$$[J^{\mu\nu}, J^{\rho\gamma}] = i\left(g^{\nu\rho}J^{\mu\gamma} - g^{\mu\rho}J^{\nu\gamma} - g^{\nu\gamma}J^{\mu\rho} + g^{\mu\gamma}J^{\nu\rho}\right)$$

$$J^{\mu\nu} = -J^{\nu\mu}$$

$J^{ik} \to \vec{J}$ rotation

$(i,k \Rightarrow 1,2,3)$, $J^{k0} = N^k$ boosts

$\Rightarrow$ $[J_i, J_j] = i\epsilon_{ijk}J_k$, $[N_i, N_j] = -i\epsilon_{ijk}J_k$

$[J_i, N_j] = -i\epsilon_{ijk}N_k$

Note: $\vec{N} = (N^1, N^2, N^3)$

Define:

$$J^{\pm} \equiv \frac{\vec{J} \pm i\vec{N}}{2}$$

$$[J^{\pm}_i, J^{\pm}_j]_- = i\,\varepsilon_{ijk}\,J^{\pm}_k$$

$$[J^+_i, J^-_j] = 0$$

$$J^{\pm\,2} = (j^{\pm})(j^{\pm}+1)$$

$2j^{\pm}+1 =$ dim matrix

$j^{\pm}$ any integer or ½ integer

Lorentz Group representation is the direct product of 2 spin group.

Representation of Lorentz Group are labeled by reps of 2 spin groups

$J_z = J^+_3 + J^-_3$

spin groups $\rightarrow (j^+, j^-)$

possible combinations

$(0,0) \Rightarrow$ systems with 0 spin

$\hookrightarrow$ 1 component

$\hookrightarrow$ Scalar field theory

$\rightarrow (\frac{1}{2}, 0), (0, \frac{1}{2}) \Rightarrow$ spin $\frac{1}{2}$ $\rightarrow$ dimension 2 reps.

Under reflection of all 3 spatial coordinates,
$J^{\pm} \rightarrow J^{\mp}$

But if we want to write systems invariant under reflections
$(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$

$\rightarrow (\frac{1}{2}, \frac{1}{2}) \Longrightarrow A^{\mu}$ (interesting!)

$(1, 0) \quad (0, 1)$

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$$(g) = \begin{pmatrix} 1 & & & \\ & -1 & & 0 \\ & 0 & -1 & \\ & & & -1 \end{pmatrix} \quad , \quad (L)^T (g) (L) = (g)$$

$$\det (g) = \pm 1$$

$$(L) = (1) + \varepsilon_{\mu\nu} (J^{\mu\nu})$$

6 of these, since $\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$

$$\Rightarrow (J^{\mu\nu}) = -(J^{\nu\mu})$$

$$[J^{\mu\nu}, J^{\rho\delta}] = i \left(g^{\nu\rho} J^{\mu\delta} + \text{permutations} \right)$$

Problem)

different $\bar{J}^{\mu\nu} = -i \left(x^{\mu} \partial^{\nu} - x^{\nu} \partial^{\mu} \right)$

Prove that $[\bar{J}^{\mu\nu}, \bar{J}^{\rho\delta}] = i \left(g^{\nu\rho} \bar{J}^{\mu\delta} + \text{permutations} \right)$

(gives an infinite dimensional representation of Lorentz group)

$$(J^{\mu\nu})_{\alpha\beta} = i \left(\delta^{\mu}_{\cdot\alpha} \delta^{\nu}_{\cdot\beta} - \delta^{\mu}_{\cdot\beta} \delta^{\nu}_{\cdot\alpha} \right)$$

$$\left.\begin{array}{lll} J^{ik} \to \vec{J} & & \text{rot.} \\ J^{k0} \to N^{k} & & \text{boost} \end{array}\right\}$$

check commutation relations

$$J^{\pm} = \frac{\vec{J} \pm i \vec{N}}{2} \quad , \quad [J^{\pm}_i, J^{\pm}_j] = i \varepsilon_{ijk} J^{\pm}_k$$

$$[J^{+}_i, J^{-}_j] = 0$$

$$J_x = -i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad \text{etc,}$$

$$N_x = -i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X' = LX$$

$$MX' = MLX \quad , \quad M^{-1}M = I$$

$$(MX') = MLM^{-1}(MX)$$

$$L = I + i\varepsilon_{\mu\nu} J^{\mu\nu} \Rightarrow MLM^{-1} = 1 + i\varepsilon_{\mu\nu}\left(MJ^{\mu\nu}M^{-1}\right)$$

$$[MJ^{\mu\nu}M^{-1}, MJ^{\alpha\beta}M^{-1}] = M[J^{\mu\nu}, J^{\alpha\beta}]M^{-1}$$

$$M[J^{\mu\nu}, J^{\rho\delta}]M^{-1} = ig^{\nu\rho}MJ^{\mu\delta}M^{-1} + \text{perm}$$

if we define $J_M^{\mu\nu} = MJ^{\mu\nu}M^{-1}$

$$[J_M^{\mu\nu}, J_M^{\rho\delta}] = ig^{\nu\rho}J_M^{\mu\delta} + \text{perms}$$

Problem: Show that under Lorentz transformations ~~that~~ that the generators of the LG, $J_{\mu\nu}$, behave like vectors

i.e. $L^T J^{\mu\nu} L = l^{\mu}_{\cdot\alpha}\, l^{\nu}_{\cdot\beta}\, J^{\alpha\beta}$

(Finished down)

Representations of LG. designated by two numbers, j^+, j^-

↳ (j^+, j^-)

↳ forms a representation of $(2j^++1)(2j^-+1)$

associated with this, $J_3 = J_3^+ + J_3^-$

$\uparrow$ measurable spin, ← momentum

$(0,0) \rightarrow$ 1 comp, scalar field

$(\frac{1}{2}, 0) \rightarrow$ spin $\frac{1}{2}$, not reflection invariant (neutrinos)

$\rightarrow$ 2 components

$(\frac{1}{2}, 0) + (0, \frac{1}{2})$ 4 comp, reflection invariant (p, n)

$\rightarrow$ reducible representation of Lorentz Group

Problem: What representation of the Lorentz Group, (j^+, j^-) is the coordinate representation we have used to define LG? Show this explicitly by examining the matrices.

Note, another coordinate transformation $'x^\mu = x^\mu + a^\mu$

a^μ is a constant 4 vector

$$P_x \equiv i\frac{\partial}{\partial x}, \quad f(x+a) = f(x) + a\frac{\partial f}{\partial x}$$

$$J^{\mu\nu} = x^\mu \partial^\nu - x^\nu \partial^\mu, \quad P_\mu \equiv i\frac{\partial}{\partial x^\mu}$$

Problem: Show the following

show that $[P_\mu, P_\nu] = 0, \quad [P_\mu, J_{\rho\delta}] = i(g_{\mu\rho}P_\delta - g_{\mu\delta}P_\rho)$

This means P_μ transforms as a 4 vector

$$\Rightarrow \overbrace{J^{\mu\nu}, P^{\mu}}^{\text{10 generators}} \text{ are generators of a lie group}$$

The group of LT + translations

Poincaré group

2N particles, assume, $m_i = m$

$$S = \sum_{i=-N}^{N} \int dt \left[p_i \frac{dq_i}{dt} - \frac{p_i^2}{2m_i} - H(p_i, q_j) \right]$$

$$S = \int \sum_{i=-N}^{N \to \infty} \left[(\Delta\epsilon) \left(\frac{p_i}{(\Delta\epsilon)^{1/2}} \right) \left(\frac{d \frac{q_i}{\sqrt{\Delta\epsilon}}}{dt} \right) - \frac{\left(\frac{p_i}{\sqrt{\Delta\epsilon}} \right)^2}{2m} - \mathcal{H} \right]$$

$$\frac{p_i(t)}{\sqrt{\Delta\epsilon}} = \mathcal{P}(t, (\Delta\epsilon)_i) \xrightarrow{\Delta\epsilon \to 0} \mathcal{P}(t, x), \quad x = \Delta\epsilon\, i$$

$$\frac{q_i(t)}{\sqrt{\Delta\epsilon}} = g(t, (\Delta\epsilon) i) \to g(t, x)$$

$$S = \int_{-\infty}^{\infty} dt\, dx \left[\mathcal{P}(t,x)\, \dot{g}(t,x) - \frac{\mathcal{P}^2(t,x)}{2m} - \mathcal{H}(\mathcal{P}, g) \right]$$

$$S = \int dt\, dx \left[\mathcal{P}(t,x) \frac{d g(t,x)}{dt} - \mathcal{H}(\mathcal{P}, g) \right]$$

$\mathcal{H}(\varphi, g)$,
actually, it may be

$$\mathcal{H}\left(\varphi, g, \frac{d\varphi}{dx}, \frac{dg}{dx} \ldots \frac{d^2 g}{dx^2} \ldots\right)$$

$$\frac{dg}{dx} = \frac{g(t, (i+1)\epsilon) - g(t, i\epsilon)}{\epsilon} \xRightarrow{\text{dim reduction}} 0$$

29/9/2005

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Probs: Due on Friday

$$S = \Sigma \int \left(p_i \dot{q}_i - \frac{p_i^2}{2m} - H(p_i, q_j)\right) dt$$

Note: When $i, j > 1$ form combinations like $p_1 p_2$, $p_2 q_1$

$\xrightarrow{\qquad}$ $$S = \int \sum_{i=1}^{N=\infty} \Delta\epsilon \frac{p_i}{\sqrt{\Delta\epsilon}} \frac{\left(\frac{dq_i}{\sqrt{\Delta\epsilon}}\right)}{dt} - \left(\frac{p_i}{\sqrt{\Delta\epsilon}}\right)^2 \cdot \frac{1}{2m} - H(p_i, q_j)$$

let particle
go to infinity

$$p_i(t) \frac{\cancel{\mathcal{P}(t, \Delta\epsilon i)}}{\sqrt{\Delta\epsilon}} \cong \mathcal{P}(t, \Delta\epsilon i) \xrightarrow[\Delta\epsilon \text{ small}]{} \mathcal{P}(t, x)$$

$\Delta\epsilon_1$, $\Delta\epsilon$

$t=0$

$$\rightarrow S = \int dt\, dx \left[\mathcal{P}(t, x)\, \dot{q}(t, x) - \frac{\mathcal{P}^2(t, x)}{2m} - H(\mathcal{P}, q)\right]$$

QM of Many Particle Sys

$$[p_i(t), q_j(t)] = -i\delta_{ij}$$

$$\epsilon \left(\left[\frac{p_i(t)}{\sqrt{\epsilon}}, \frac{q_j(t)}{\sqrt{\epsilon}} \right] = -i \frac{\delta_{ij}}{\epsilon} \right)$$

$$\sum_i \epsilon [\mathcal{P}(t, x_i), q(t, x_j)] = -i \sum_i \delta_{ij} = -i$$

Limit $\epsilon \to 0$ $\downarrow$

$$\int dx \left[\mathcal{P}(t, x), q(t, y)\right] = -i$$

$$\Rightarrow \boxed{[\mathcal{P}(t, x), q(t, y)] = -i\delta(x - y)}$$

$\mathcal{H} + \frac{p^2}{2m}$

Expect

$$\int_{\text{all } x} dx\, \mathcal{H}(t,x) = H \quad , \text{ and that } \frac{d}{dt}H = 0$$

$$S = \int dt\, d^m x \left[\mathcal{P}(t,\vec{x})\, \dot{g}(t,\vec{x}) - \frac{\mathcal{P}^2(t-\vec{x})}{2m} - \mathcal{H}(\mathcal{P}, g) \right]$$

$m = \#$ of spatial dim. $\qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$

m spatial dimensions, generalize LG where $S = X^{0^2} - X^{1^2} - X^{2^2} - \cdots X^{m^2}$

$= S' = {}'X^{0^2} - {}'X^{1^2} - {}'X^{2^2} - \cdots {}'X^{m^2}$

$\Rightarrow$ insist that this is Lorentz invariant

further require that there is only one canonical momentum $\mathcal{P}(t,x)$

$\Rightarrow$ New names Call $\mathcal{P}(t,\vec{x}) = \phi^0(t,\vec{x})$

also call $g(t,\vec{x}) = \phi(t,\vec{x}) = \phi(x)$

$$S = \int d^n x \left[\phi^0(x) \frac{d\phi}{dt} - \mathcal{H}(\phi, \phi^0, \ldots) \right]$$

$n = m+1$

PROBLEM

Prove that under proper LT, $d^n x = d^n x'$

$$S = \int d^n x \left[\phi^\mu \frac{\partial \phi}{\partial x^\mu} - \mathcal{H}_2(\ ) \right]$$

$$\phi^\mu \frac{\partial \phi}{\partial x^\mu} = \phi^0 \frac{\partial \phi}{\partial x^0} + \phi^k \frac{\partial \phi}{\partial x^k}$$

I have m new fields ϕ^k

ϕ assumed scalar

$$\mathcal{H}_2 = H_2(\phi^\mu, \phi)$$

$$\hookrightarrow H_2(\phi^\mu \phi_\mu, \phi)$$

guess simple stuff that is Lorentz invariant

$$S = \int d^n x \left[\phi^\mu \partial_\mu \phi + a\phi + \frac{b}{2}\phi^\mu \phi_\mu + \frac{c}{2}\phi^2 + \cancel{a\ \text{mess}} \right]$$

QFT of a free scalar

Note: for the time being, ignore $a\phi$: .

reason: define $\phi' = \phi + e$

then substitute ϕ' for ϕ in the S

PROBLEM: Show that $S(\phi', \phi^\mu) = S(\phi, \phi^\mu)$

If this is a quantum field theory:

we know that $[\phi^0(t, \bar{x}), \phi(t, \vec{y})] = \cancel{-i\delta(\bar{x}-\vec{y})} = -i\delta^n(\bar{x}-\vec{y})$

δS, treat ϕ^μ and ϕ as independent in the action

$$\delta S = \int d^n x \left(\delta\phi^\mu [\partial_\mu \phi + b\phi_\mu] + \delta\phi[-\partial_\mu \phi^\mu + c\phi] \right)$$

$$+ \underbrace{\int d^n x\, \partial_\mu (\phi^\mu \delta\phi)}_{\text{surface term}}$$

$$\delta\phi^\mu \rightarrow \partial_\mu \phi + b\phi_\mu = 0$$

$$\delta\phi \rightarrow -\partial_\mu \phi^\mu + c\phi = 0$$

PROBLEM: Assume that under Lorentz Transformation $\phi'(x') = \phi(x)$

then show that $\partial_\mu \phi(x) = \frac{\partial}{\partial x^\mu}\phi(x)$ transforms like a vector

Reminder: $'X^\mu = \ell^\mu{}_\nu X^\nu$

$$\phi_\mu = -\frac{1}{b} \partial_\mu \phi$$

$$-\partial_\mu \left[-\frac{1}{b} \partial^\mu \phi \right] + c\phi = 0$$

$$\frac{1}{b} \partial_\mu \partial^\mu \phi + c\phi = 0 \quad , \quad (\partial_\mu \partial^\mu)\phi + bc\phi = 0$$

$$bc = m^2$$

$$\boxed{(\partial^2 + m^2)\phi = 0} \quad \text{Klein - Gordon Equation}$$

Use standard form

$$S = \int d^n x \left[\phi^\mu \partial_\mu \phi - \frac{\phi^\mu \phi_\mu}{2} - \frac{m^2}{2}\phi^2 \right]$$

by the way $\Rightarrow$ $\partial_\mu \phi = \phi_\mu$

$$-\partial_\mu \phi^\mu = m^2 \phi$$

$$\Leftrightarrow (\partial^2 + m^2)\phi = 0$$

$$\Rightarrow S = \int d^n x \left[\frac{\partial^\mu \phi \partial_\mu \phi}{2} - \frac{m^2 \phi^2}{2} \right]$$ $\Rightarrow$ This is in the books!

$$\phi(x) = \int d^n k \, e^{ik^\mu x_\mu} \phi(k)$$

$$\int dk \left(-k^2 + m^2 \right) e^{ikx} \phi(k) = 0$$

$$\Rightarrow (-k^2 + m^2)\phi(k) = 0$$

If $\phi(k) \neq 0$ $\Rightarrow$ $-k^2 + m^2 = 0$

$$-k^{0\,2} + (\vec{k})^2 + m^2 = 0$$

$$\Rightarrow k^0 = \pm (\vec{k}^2 + m^2)^{1/2}$$

10/4/2005

Summary

$$\int dt \left[P \frac{dq}{dt} - H \right] \Rightarrow S = \int d^n x \left[\phi^0(\vec{x},t) \frac{d\phi(\vec{x},t)}{dt} - \mathcal{H}(\phi, \phi^0 \_) \right]$$

$$H \equiv \int d^n x \, \mathcal{H}(-)$$

$$QM \quad i[\phi^0(\vec{x},t), \phi(\vec{y},t)] = \delta^n(\vec{x}-\vec{y})$$

Assume that we will only deal with manifestly Lorentz invariant actions.

Look for simples Assume $\phi(\vec{x},t) \equiv \phi(x)$ is a scalar
$\Rightarrow \phi'(x') = \phi(x)$

$$\int d^n x \left[\phi^\mu \partial_\mu \phi - \mathcal{H}'(\phi, \phi^0 \_) \right.$$

To do this require that $\mathcal{H}$ contains the n fields ϕ'
Postulate that $\phi^\mu(x')$ is a Lorentz n vector

$$\phi'^\mu(x') = \ell^\mu{}_{,\nu} \phi^\nu(x)$$

$$S = \int d^n x \left[\phi^\mu \partial_\mu \phi - \mathcal{H}' \cancel{(\phi^\mu \phi_\mu, \ldots)} \right]$$

$$\mathcal{H}' = \mathcal{H}'\left(g(\phi_\mu \phi^\mu), f(\phi) \right]$$, f & g are any function.

$$S = \int d^n x \left\{ \left[\phi^\mu \partial_\mu \phi - \frac{1}{2} \phi^\mu \phi_\mu - \frac{m^2}{2} \phi^2 \right] - \mathcal{H}''\left(\mathcal{H}(\phi), \mathcal{I}(\phi^\mu \phi_\mu) \right) \right\}$$

$\mathcal{H}$ & $\mathcal{I}$ arbitrary functions

$\delta S = 0$

$$\left.\begin{aligned} &\delta \phi^\mu \longrightarrow \partial_\mu \phi = \phi_\mu \\ &\delta \phi \rightarrow -\partial_\mu \phi^\mu - m^2 \phi = 0 \end{aligned}\right) \Rightarrow \left(-\partial^2 - m^2\right) \phi = 0$$

Two time derivative action i.e get rid of ϕ^μ

$$S = \int d^n x \left[\frac{1}{2} (\partial^\mu \phi)(\partial_\mu \phi) - \frac{m^2}{2} \phi^2 \right]$$

$$S = \int d^n x \left[\phi^0 \partial_0 \phi + \underbrace{\left\{ \phi^k \partial_k \phi - \frac{\phi^0 \phi_0}{2} - \frac{\phi^k \phi_k}{2} - \frac{m^2}{2} \phi^2 \right\}}_{-\mathcal{H}} \right]$$

$$H = \int d^n x \left[-\phi^k \partial_k \phi + \frac{(\phi^0)^2}{2} + \frac{\phi^k \phi_k}{2} + \frac{m^2 \phi^2}{2} \right]$$

$$\phi_k = \frac{\partial}{\partial x^k} \phi \quad , \quad \phi^k = g^{k\mu} \phi_\mu = -\phi_k$$

$$\Rightarrow \quad H = \int d^n x \left(\partial_k \phi^2 + \frac{(\phi^0)^2}{2} - \frac{(\partial_k \phi)^2}{2} + \frac{m^2 \phi^2}{2} \right)$$

$$H = \int d^m x \left[\frac{\vec{\nabla} \phi \cdot \vec{\nabla} \phi}{2} + \frac{(\partial_0 \phi)^2}{2} + \frac{m^2 \phi^2}{2} \right]$$

~~Assume~~

Assume that this quantum mechanics is described by a complete set of states. having energy eigenvalues E_i and written as

$$|E_i\rangle \text{ then } \boxed{H|E_i\rangle = E_i|E_i\rangle}$$

$E_i \geq 0$. Assume there is a state of lowest energy which we call the vacuum state: We assume that the energy of the vacuum state can be renormalized to zero.

Problem) Dimensionally reduce our scalar action to zero spatial dimensions, Relate this to the harmonic oscillator

Vacuum state is designated by $|0\rangle$

Problem) Using the equations of motion prove that $\frac{dH}{dt} = 0$.

For this problem, assume that all fields $\to 0$ at spacial infinity

Looking for $T^{\mu\nu}$ $\rightarrow$ which are the generators of the Lorentz Group.

$H = \int d^m x\, \mathcal{H}$, $\frac{dH}{dt} = 0$, $P^k = \int d^m x\, \mathcal{P}^k$

$$\frac{d}{dt} P^k = 0$$

Noether's Theorem $\quad H = \frac{1}{2}\left[(\partial^0\phi)^2 + (\partial_k \phi)^2 + m^2\phi^2\right]$

for conserved $P = \int d^3x\, \mathcal{P}$, we need $\frac{d\mathcal{P}}{dt} = 0$

question Can we write down a lorentz vector density $= J^\mu$;

$$\partial_\mu J^\mu = 0$$

Note if $\cancel{\frac{d}{dt}}\ \frac{\partial}{\partial x^\mu} J^\mu = 0$, $\Rightarrow \int_{\text{all space}} d^m x \frac{\partial}{\partial x^\mu} J^\mu$

$$= \frac{d}{dt}\int d^n x\, J^0 + \underbrace{\int d^n x\, \vec{\nabla}\cdot\vec{J}}_{\cancel{\int_{\text{surf}} J\cdot da}}$$

assume fields vanish at spacial infinity

~~T~~ Look for $T^{\mu\nu}$ such that $\partial_\mu T^{\mu\nu} = 0$ and $\int d^3x\, T^{00} = H$

because $H = (\phi^0)^2 + (\nabla\phi)^2 + \frac{m^2}{2}\phi^2$

If these conditions are met, then

$$p^k = \int d^n x \, T^{0k}$$

$$\partial_\mu T^{\mu\nu} = 0 \longrightarrow \int d^n x \, \partial_\mu T^{\mu\nu} = 0$$

$$\Rightarrow \partial_0 \int d^n x \, T^{0\nu} = 0$$

$$\Rightarrow n \text{ quantities } \boxed{p^\nu \equiv \int d^n x \, T^{0\nu}} \ \ni: \frac{d}{dt} p^\nu = 0$$

Candidates for $T^{\mu\nu}$

$$T^{\mu\nu} = c\, \phi^\mu \phi^\nu + g^{\mu\nu} \left[b\, \phi^\mu \phi_\mu + a \phi^2 \right]$$

$$T^{00} = \frac{(\partial_0 \phi)^2 + \nabla\phi^2 + m^2 \phi^2}{2}$$

$$T^{00} = c\, \phi^{0\,2} + b\left[\phi^{0\,2} - (\partial_k \phi)^2\right] + a\phi^2$$

$$a = \frac{m^2}{2}, \quad c + b = \frac{1}{2}, \quad b = -\frac{1}{2}, \quad c = 1$$

~~T^{00}~~ $\Rightarrow$ $$\boxed{T^{\mu\nu} = \phi^\mu \phi^\nu + g^{\mu\nu}\left[-\frac{1}{2}\phi^\alpha \phi_\alpha + \frac{m^2}{2}\phi^2\right]}$$

Prove: Prove that $\boxed{\partial_\mu T^{\mu\nu} = 0}$

$\Rightarrow$ because $\partial_\mu T^{\mu\nu} = 0$

That $\frac{d}{dt} P^\nu = 0$, where $P^\nu = \int d^m x \, T^{0\nu}$

$T^{\mu\nu} = T^{\nu\mu}$, $\boxed{R^{\lambda\mu\nu} \equiv X^\mu T^{\lambda\nu} - X^\nu T^{\lambda\mu}}$

$\partial_\lambda R^{\lambda\mu\nu} = \delta^\mu_\lambda T^{\lambda\nu} + 0 - \delta^\nu_\lambda T^{\lambda\mu} + 0$

$= T^{\mu\nu} - T^{\nu\mu} = 0$

10/6/2005

All space-time

$$S_\phi = \int d^n x \left[\phi^\mu(x) \partial_\mu \phi(x) - \frac{1}{2} \phi^\mu(x) \phi_\mu(x) - \frac{m^2}{2} \phi^2 \right]$$

$$(\partial^2 + m^2) \phi = 0$$

Under LT $\phi(x) \to \phi'(x')$

$\phi'(x') = \phi(x)$ → scalar field

$$S = S' = \int d^n x' \left[\phi'^\mu(x') \partial_\mu \phi'(x' \_\_ \right]$$

$$S_A = \int d^n x \left[-F^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{F^{\mu\nu} F_{\mu\nu}}{2} \right]$$

$$\delta S \Rightarrow \delta F \left(-(\partial_\mu A_\nu - \partial_\nu A_\mu) + F_{\mu\nu} \right) = 0$$

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}$$

$$\delta A \quad \boxed{\partial_\mu F^{\mu\nu} = 0} \quad \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = 0$$

$$\partial^2 A^\nu - \partial^\nu (\partial \cdot A) = 0$$

$A'_\mu = A_\mu + \partial_\mu \lambda(x)$ → gauge transformation

$\partial_\mu A^\mu = \partial \cdot A = 0$, Lorentz gauge

$$\Rightarrow \partial^2 A^\nu = 0$$

$$\underset{\phi, I}{S} = \int_{\substack{\text{All} \\ \text{space-} \\ \text{time}}} d^n x \left[\phi^\mu(x)\, \partial_\mu \phi(x) - \frac{1}{2}\phi^\mu(x)\,\phi_\mu(x) - \frac{m^2}{2}\phi^2 - H_I(\phi, \phi^\mu \phi_\mu) \right]$$

By the rules we develop, we will find that $H_I(\phi, \phi^\mu\phi_\mu) \overset{n \geq 4}{=} b\phi^3 + \frac{g\phi^4}{4} + c\phi$

is the only thing allowed.

for this course $H_I = H_I(\phi)$

$$S = \int dt \left\{ P(t)\, \frac{dQ}{dt} - H \right\}$$

$$H = \int \left[\frac{1}{2}\left[(\partial^0\phi)^2 + (\nabla\phi)^2 + \frac{m^2\phi^2}{2} \right] + H_I(\phi) \right] d^n x$$

$|a'\rangle$, a' = energy eigenvalues

$H|0\rangle = 0\,|0\rangle$, $|0\rangle \equiv$ vacuum $\equiv$ state of lowest energy

$$T^{\mu\nu}(x) \;;\; \partial_\mu T^{\mu\nu} = 0 \Rightarrow \frac{d}{dt}\int d^n x\, T^{0\nu} = 0$$

$$\Rightarrow P^\nu = \int d^n x\, T^{0\nu}$$

$$J^{\lambda\mu\nu} \equiv x^{\mu}T^{\lambda\nu} - x^{\nu}T^{\lambda\mu}$$

$$\partial_\lambda J^{\lambda\mu\nu} = \delta^{\mu}_{\lambda}\, T^{\lambda\nu} + \underbrace{x^{\mu}\,\partial_\lambda T^{\lambda\nu}}_{0} - \delta^{\nu}_{\lambda}\, T^{\lambda\mu} - \underbrace{x^{\nu}\,\partial_\lambda T^{\lambda\mu}}_{0}$$

$$= T^{\lambda\mu} - T^{\nu\mu}$$

<u>Note:</u> If $T^{\mu\nu} = T^{\nu\mu}$ then $\boxed{\partial_\lambda J^{\lambda\mu\nu} = 0}$

require $T^{00} = \frac{1}{2}\left(\phi^{0\,2} + (\nabla\phi)^2 + m^2\phi^2\right)$

<u>Notes</u> $\int d^3x\, T^{00} \Rightarrow H$

$$T^{\mu\nu} \equiv \phi^{\mu}\phi_{\nu} - \frac{g^{\mu\nu}}{2}\left(\phi^{\mu}\phi_{\mu} - m^2\phi^2\right)$$

<u>Problem:</u> Prove that $\partial_\mu T^{\mu\nu} = 0$ using E.O.M
$(\partial^2 + m^2)\phi = 0$

1. Implicit in all of this is that $[\delta\phi, \phi] = 0$, $[\delta\phi^{\mu}, \phi^{\nu}] = 0$
$[\delta\phi, \phi^{\mu}] = 0$

2. $\phi^{\mu}\phi^{\nu}$ means in QFT
$$\frac{\phi^{\mu}\phi^{\nu} + \phi^{\nu}\phi^{\mu}}{2}$$

Noether's Theorem

Sym $\longrightarrow$ generator (of the Lie group)

$$S = \int_a^b d^n x \, \mathcal{L}(\phi, \phi^{\mu})$$

finite volume in space-time

$$S' = \int_{a'}^{b'} d^n x' \, \mathcal{L}(\phi'(x'))$$

$$S' = S$$

$$\Delta S = S' - S$$

Translation, rotations etc-

$$\phi(x) \rightarrow \phi'(x') \quad , \quad \phi'(x') = \phi(x)$$

$$0 = \phi'(x') - \phi(x) = [\phi'(x') - \phi'(x)] + [\phi'(x) - \phi(x)]$$

$$\phi'(x') - \phi'(x) = \delta x^{\mu} \frac{\partial \phi'}{\partial x^{\mu}}(x), \quad \delta x^{\mu} = x' - x$$

$$\delta x \frac{\partial \phi'}{\partial x}(x) = \cancel{} \; \alpha \, \delta x^{\mu} \frac{\partial \phi(x)}{\partial x^{\mu}} + O(\delta x^2)$$

$$0 = \delta x^{\mu} \frac{\partial \phi}{\partial x^{\mu}} + \underbrace{\phi'(x) - \phi(x)}_{\delta \phi(x)} \quad , \quad \boxed{\delta \phi(x) = - \delta x^{\mu} \frac{\partial \phi}{\partial x^{\mu}}}$$

$$\int_{a+\delta a}^{b+\delta b} dt\, f(t) - \int_a^b dt\, f(t) = \delta b\, f(b) - \delta a\, f(a)$$

$$\delta \int_a^b dt\,[f(t)] \rightarrow \int_a^b \delta(dt)\, f(t) = \int_a^b dt \left[\frac{\partial}{\partial t}\delta t\right] f(t)$$

variation
of end point

$$S = \int dx\, \mathcal{L}(x)$$

$$\Delta S = \int \delta(dx)\, \mathcal{L}(x) + \int dx\, (\Delta\mathcal{L})$$

$$\Delta S = \int dx \left[\frac{\partial}{\partial x^\mu}\delta x^\mu\right] \mathcal{L}(x) + \int dx\, \underbrace{\left[\mathcal{L}'(x') - \mathcal{L}(x)\right]}_{\delta\mathcal{L} + \frac{\partial \mathcal{L}}{\partial x^\mu}\delta x^\mu}$$

0 by field equations

<u>Remember</u>, $$\delta\mathcal{L} = \delta\phi\left[-\partial_\mu \phi^\mu - m^2\phi\right] + \delta\phi\left[\partial_\mu\phi - \phi_\mu\right] + \frac{\partial}{\partial x^\mu}\left[\phi^\mu \delta\phi\right]$$

$$\Delta S = \int dx \left[\frac{\partial}{\partial x^\mu}\left[\phi^\mu \delta\phi\right] + \frac{\partial}{\partial x^\mu}\left[\mathcal{L}\,\delta x^\mu\right]\right]$$

$$\Delta S = \int dx\, \frac{\partial}{\partial x^\mu}\left[\phi^\mu \delta\phi + \mathcal{L}\,\delta x^\mu\right]$$

$$\Delta S = 0$$

$$\Delta S = \int dx\, \frac{\partial}{\partial x^\mu}\left[-\delta x^\nu \frac{\partial\phi}{\partial x^\nu}\phi^\mu + \mathcal{L}\,\delta x^\mu\right]$$

Sort this out

First look just at translation $\delta X^\mu = \varepsilon^\mu$, [$\varepsilon^\mu$ is a set of small numbers]

Most general $\delta X^\mu = \varepsilon^{\mu}_{\cdot\nu} X^\nu + \varepsilon^\mu$

$$\Delta S \xrightarrow[\text{for translations}]{} \int dx \frac{\partial}{\partial x^\mu}\left[-\phi^\mu \phi_\nu + \mathcal{L}\,\delta^\mu_{\cdot\nu}\right]\varepsilon^\nu = 0$$

$$\Rightarrow \cancel{\frac{\partial}{\partial x^\mu} T^{\mu\nu}} \quad -\frac{\partial}{\partial x^\mu} T^\mu_{\ \nu} = 0$$

$$\boxed{T^\mu_{\cdot\nu} = -\phi^\mu \phi_\nu + \delta^\mu_\nu\left[\frac{1}{2}\phi^\alpha \phi_\alpha - \frac{m^2\phi^2}{2}\right]}$$

This is exactly the $T^\mu_{\cdot\nu}$ we guessed

Problem Show by examining $\delta X^\mu = \varepsilon^\mu_{\cdot\nu} X^\nu$, [you get the other 6 generators]

9/11/2005

$$\delta X^\mu = \varepsilon^\mu{}_\nu X^\nu$$

$$\partial_\lambda J^{\lambda\mu\nu} = 0 \,, \quad J^{\lambda\mu\nu} = X^\mu T^{\lambda\nu} - X^\nu T^{\lambda\mu}$$

$$\Rightarrow J^{\mu\nu} \equiv \int d^n x \left[X^\mu T^{0\nu} - X^\nu T^{0\mu} \right]$$

has property $\frac{d}{dt} J^{\mu\nu} = 0$

Facts, $p^\mu \to (H, p^k)$

$$i[H, \phi(x)] = i\left[\int d^n x' \, T^{00}(x', t), \phi(x, t)\right]$$

$$H = \int d^n x' \left[\frac{\phi^{0\,2} + \phi^{k\,2} + m^2 \phi^2 (x', t')}{2} \right] \,, \quad \text{because } \frac{dH}{dt} = 0$$

$$i\left[\int d^n x' \, \frac{\phi^{0\,2}(x', t)}{2}, \phi(x, t) \right] = \phi^0(x, t) = \frac{\partial}{\partial x^0} \phi(x, t)$$

~~Proble~~ Problem, Show that if $\mathcal{L} = \phi^\mu \partial_\mu \phi + \frac{\phi^\mu \phi_\mu}{2} - \frac{m^2 \phi^2}{2} - \frac{H(\phi)}{I}$

$$i[H, \phi] = \frac{\partial \phi}{\partial x^0}$$

$$i[p^k, \phi(x)] = i\left[\int d^3 x \, [\phi^k \phi^0, \phi(x)\right] = \phi^k(x) = \frac{-\partial \phi}{\partial x^k}$$

$$i[P^\mu, \phi(x)] = \partial^\mu \phi(x)$$

~~[illegible]~~

Problem) part b) Write down P^k when $H_I(\phi) \neq 0$

Thus answer $\Rightarrow$ above is true even with interaction

Problem) Prove: $[P^\mu, P^\nu] = 0$

for free field i.e. $H_I(\phi) = 0$

Remarks $[T^{00}(x', t), T^{00}(x, t)] \neq 0 \propto T^{0k}(x)\, \partial_k \delta^m(x - x')$

$$\boxed{\phi(x) = e^{iP^\mu x_\mu} \phi(0)\, e^{-iP^\nu x_\nu}}$$ Exact even if $H_I(\phi) \neq 0$

$$\frac{\partial}{\partial x^\alpha} e^{iP^\mu x_\mu} = i P_\alpha e^{iP^\mu x_\mu}$$

$$-i[J^{\mu\nu}, J^{\lambda\sigma}] = g^{\mu\lambda} J^{\nu\sigma} - g^{\nu\lambda} J^{\mu\sigma} - g^{\mu\sigma} J^{\nu\lambda} + g^{\nu\sigma} J^{\mu\lambda}$$

$\downarrow$

$$\int d^m x \left[x^\mu T^{0\nu} - x^\nu T^{0\mu} \right] =$$

$\Rightarrow$ $J^{\mu\nu}$ are generators of Poincare group $\Rightarrow$ ~~[illegible]~~ describe a representation of that group.

$J^{\mu\nu}$ are made of quantum operators $\rightarrow$ state vectors are $|\alpha\rangle$

Special Problem) (Prize: a cup of coffee of your choice $\leq \$5$)

Prove these (all of them)

$$[J^{\mu\nu}, \phi] =$$

Problems prove $\rightarrow$ $$i[J^{12}, \phi(x,t)] = \left[-x^1 \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial x^1} \right] \phi$$

$$i[J^{\mu\nu}, \phi] = (x^\mu \partial^\nu - x^\nu \partial^\mu)\, \phi$$

Solve free scalar field theory

Assume set of states

$$(\partial^2 + m^2)\phi(x,t) = 0 \quad , \quad i[\partial^0 \phi(\bar{x},t), \phi(\bar{y},t)] = \delta^n(\bar{x}-\bar{y})$$

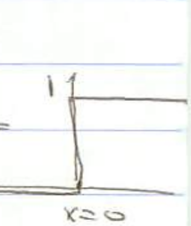
Assume there is a state of lowest energy $|0\rangle$.
do this in 4 dimensions

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik^\mu x_\mu}\, \phi(k)$$

Note that $\phi(k)$ is an operator

$$(-k^\mu k_\mu + m^2)\, \phi(k) = 0$$

$$\phi(k) = \delta(-k^2 + m^2)\, a(k) \quad , \quad [\text{ie } x\,\delta(x) = 0]$$

$\Theta(x) =$ 

$$\frac{d\Theta}{dx} = \delta(x)$$

$$\phi(k) = \delta(-k^{0\,2} + \bar{k}^2 + m^2)\, a(k)$$

$$= \left[\frac{\delta(k^0 + (\bar{k}^2+m^2)^{1/2})}{2(\bar{k}^2+m^2)^{1/2}}\, a(k)\, \Theta(-k^0) + \frac{\delta(k^0 - (\bar{k}^2+m^2)^{1/2})}{2(\bar{k}^2+m^2)^{1/2}}\, a(k)\, \Theta(k^0) \right.$$

Rewrite yet again

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4} \, e^{ik^\mu x_\mu} \, \phi(k)$$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{[2(k^2+m^2)^{1/2}]^{1/2}} \left[e^{ik^\mu x_\mu} a^+(k) + e^{-ik^\mu x_\mu} a^-(k) \right]$$

Note $k^0 = +\sqrt{k^2+m^2}$

$$a^+(k) \equiv \frac{a(k)\,\theta(k^0)}{(2k^0)^{1/2}\,(2\pi)^{1/2}}$$

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ϕ is an operator

since $S^\dagger = S$, choose $\phi^\dagger = \phi$

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{ik^\mu x_\mu} \phi(k)$$ ↗ operator

$$FE \Rightarrow (-k^\mu k_\mu + m^2)\phi(k) = 0$$

$$\phi(k) \equiv \delta(-k^2+m^2)\, a(k)$$

$$\downarrow$$

$$\theta(-k^0) A\, \delta(k^0 + \sqrt{k^2+m^2}) + A\,\delta(k^0 - \sqrt{m^2+k^2})\,\theta(k^0)$$

$$A = \frac{1}{2(\vec{k}^2+m^2)^{1/2}}$$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}(2k^0)^{1/2}} \left[e^{ik^\mu x_\mu} a^+(k) + e^{-ik^\mu x_\mu} a^-(k) \right]$$

$$k^0 \equiv (\vec{k}^2+m^2)^{1/2}$$

$$a^+(k) = \frac{a(k)\,\theta(k^0)}{(2k^0)^{1/2}(2\pi)^{3/2}}, \quad a^-(k) = \frac{a(-k)\,\theta(k^0)}{(2k^0)^{1/2}(2\pi)^{3/2}}$$

Just comes from a series of trivial definitions.

$\partial_0 \phi(\vec{x}, t)$

$\parallel$

$$i\,[\,\phi^\circ(x,t), \phi(\vec{x},t)\,] = \delta^3(\vec{x}-\vec{y})$$

$$[\phi(x,t), \phi(y,t)] = 0$$

Problem: Prove $\left[a^+(\vec{k}), a^-(\vec{k}')\right] = -\delta^3(\vec{k}-\vec{k}')$

$$[a^\pm(\vec{k}), a^\pm(\vec{k}')] = 0$$

Problem: Prove $H = \int \frac{d^3x}{2}\left[\phi^{\circ 2} + (\nabla\phi)^2 + m^2\phi^2\right]$

$$H = \int d^3k\,\frac{k^\circ}{2}\left[a^+(\vec{k})a^-(\vec{k}) + a^-(\vec{k})a^+(\vec{k})\right]$$

$$H = \int d^3k\,\frac{k^\circ}{2}\Big(2a^+(\vec{k})\,a^-(\vec{k}) + \underbrace{\delta^3(\vec{k}-\vec{k}^\circ)}_{\delta^3(0)}\Big)$$

$$H = \int d^3k\left[a^+(k)\,a^-(k)\,k^\circ + \underbrace{\frac{k^\circ}{2}\delta^3(0)}_{\text{Ground state energy.}}\right]$$

We will renormalize and write (the shifted Energy Hamiltonian) as

$$\boxed{H_R \equiv \int d^3k\;k^\circ a^+(\vec{k})\,a^-(\vec{k})}$$

Hypothesize state of lowest energy

$|0\rangle$, $H|0\rangle = 0$

$$p^\mu = \int d^3k\;\frac{k^\mu}{2}\left(a^+(k)\,a^-(k) + a^-(k)\,a^+(k)\right)$$

$\rightarrow$ because the integral is odd

$$p^{\ell} = \int d^3k \; k^{\ell} \left(a^+(k)\, a^-(k) + \cancel{\delta^3(0)} \right)$$

<u>assumptions</u> $|0\rangle$

$H|0\rangle = 0$

$p^{\mu}|0\rangle = 0$

$J^{\mu\nu}|0\rangle = 0$

$$[H, a^-(\ell)] = \left[\int d^3k\, k^0 \left[a^+(k)\, a^-(k), a^-(\ell)\right]\right]$$

$$= \int d^3k\, k^0 \,[a^+(k), a^-(\ell)]\, a^-(k) = -\ell^0 a^-(\ell)$$

$$[a^+(\vec{k}), a^-(\vec{k}')] = -\delta^3(\vec{k}-\vec{k}')$$

$$\Rightarrow \quad [p^{\mu}, a^-(\ell)] = -\ell^{\mu} a^-(\ell),$$

$$[p^{\mu}, a^+(\ell)] = \ell^{\mu} a^+(\ell)$$

$\Rightarrow$ Other states of this QM

$$\underbrace{\left(a^+(k_1)\, a^+(k_2)\, a^-(\ell_1)\, a^-(\ell_2) - a^+ \_\, a^-\right)}_{\text{In this manner make all states.}} |0\rangle$$

$$[p^{\mu}, a^+(\ell)]\, |0\rangle = \ell^{\mu} a^+(\ell)|0\rangle$$

$$[p^{\mu} a^+(\ell) - a^+(\ell)\, p^{\mu}]\,|0\rangle = \ell^{\mu} a^+(\ell)|0\rangle$$

$$p^{\mu}\left(a^+(\ell)|0\rangle\right) = \ell^{\mu}\left(a^+(\ell)|0\rangle\right)$$

$$\Rightarrow \quad a^{+}(\ell)|0\rangle \equiv |\ell^{\mu}\rangle$$

$$[p^{\mu}a^{-}(\ell) - a^{-}(\ell)p^{\mu}]|0\rangle = \ell^{\mu}(a^{-}(\ell)|0\rangle)$$

$$p^{\mu}(a^{-}(\ell)|0\rangle) = -\ell^{\mu}(a^{-}(\ell)|0\rangle$$

$$\Rightarrow a^{-}(\ell)|0\rangle = 0$$

$$\cancel{p^{\mu}(a^{-}(\ell)a^{+}(k)} \quad p^{\mu}(a^{-}(\ell)a^{+}(k)|0\rangle) = (k^{\mu}-\ell^{\mu})(a^{-}(\ell)a^{+}(k)|0\rangle)$$

Problem: Prove this

Observations

$$|0\rangle, |\ell_1^{\mu}, \ell_2^{\mu}, \_\_\_\rangle$$

$$p^{\mu}|\ell_1^{\mu}, \ell_2^{\mu}, \_\_\rangle = (\ell_1^{\mu}+\ell_2^{\mu}+\_\_)|\ell_1^{\mu}, \ell_2^{\mu}, \_\_\rangle$$

$$H[a_{+}({}^{1}k^{\mu})a_{+}({}^{2}k^{\mu})|0\rangle] = (k_1^{0}+k_2^{0})[a_{+}a_{+}|0\rangle]$$

$$(\partial^2+m^2)\langle 0|\phi(x)|0\rangle = 0 \qquad , \quad i[p^{\mu}, \phi(x)] = \partial^{\mu}\phi(x)$$

$$\phi(x) = e^{ip^{\mu}x_{\mu}}\phi(0)e^{-ip^{\mu}x_{\mu}}$$

$$(\partial^2+m^2)\langle 0|e^{ip^{\mu}x_{\mu}}\phi(0)e^{-ip^{\mu}x_{\mu}}|0\rangle = 0$$

$$(\partial^2+m^2)\underbrace{\langle 0|\phi(0)|0\rangle}_{\text{constant}} = 0$$

$$\boxed{m^2 \langle 0|\phi(0)|0\rangle = 0}$$

$\Rightarrow \quad m^2 = 0 \quad$ or $\quad \langle 0|\phi(0)|0\rangle = 0$

$\Downarrow$

our $\phi \sim \int a^{k}(x) + a^{-}(k)$

$\langle 0|\phi|0\rangle = 0$

we ignored this possibility.

$m^2 \neq 0 \Rightarrow \langle 0|\phi(x)|0\rangle = \langle 0|\phi(0)|0\rangle = 0$

$m^2 = 0 \qquad \langle 0|\phi(x)|0\rangle = \eta \neq 0$

$(\partial_\mu \partial^\mu + \cancel{m^2})\phi = 0$, $\quad \partial_\mu(\partial^\mu \phi) = 0$ if $m^2 = 0$

$\Downarrow$

$\int d^3x \; \partial^0 \phi(x,t) = Q$, and $\frac{dQ}{dt} = 0$

<u>Inverse Goldstone Theorem</u>

10/18/2005

$(\partial^2 + m^2)\phi = 0$

$i[\partial_0 \phi(\vec{x},t), \phi(\vec{y},t)] = \delta^3(\vec{x}-\vec{y})$

general $\rightarrow$ $i[P^\mu, \phi(x)] = \partial^\mu \phi(x)$

$|0\rangle$

$$\boxed{\begin{aligned} P^\mu|0\rangle &= 0 \\ J^{\mu\nu}|0\rangle &= 0 \end{aligned}}$$

any poincare inv. theory

since

$$\phi(x,t) = \int \frac{d^3k}{(2\pi)^{3/2}(k^0)^{1/2}}\left[e^{ik^\mu x_\mu} a^+(k) + e^{-ik^\mu x_\mu} a^-(k)\right]$$

$$k^0 \equiv +(\vec{k}^2 + m^2)^{1/2}$$

$a^-(\vec{k})|0\rangle = 0$
$a^+(\vec{k})|0\rangle = |k\rangle$, $P^\mu|k\rangle = k^\mu|k\rangle$

$a^+(k)\, a^+(k')|0\rangle = |k,k'\rangle$, $P^\mu|k,k'\rangle = (k^\mu + k'^\mu)|k,k'\rangle$

$\phi(x)$, $|0\rangle$

$\phi(x_1)\phi(x_2)$ — $\phi(x_2)|0\rangle$

$\int d^4x_1\, d^4x_2\, f(x_1,x_2)\, \phi(x_1)\phi(x_2)|0\rangle$ somehow complete

$$|k\rangle = a^+(k)\,|0\rangle$$

$$\langle k'|k\rangle = \langle 0|\,a^-(k')\,a^+(k)\,|0\rangle$$

$$[a^-(k'), a^+(k)] = \delta^3(k'-k)$$

$$\langle k'|k\rangle = \langle 0|\,[a^-(k')a^+(k) - a^+(k)a^-(k')]\,|0\rangle$$

$$= \langle 0|\,[a^-(k'), a^+(k)]\,|0\rangle = \langle 0|\,\delta^3(k'-k)\,|0\rangle$$

$$\xrightarrow[k' \to k]{} \;=\; \infty\,\langle 0|0\rangle$$

$f(k)\, a(k)\,|0\rangle$ → to work choose $f(k)$

$f(k)$ arbitrary,

Note $\langle k'|k\rangle = 0$ if $k' \neq k$

$$P^\mu, J^{\mu\nu} \rightarrow P^\mu = \int k^\mu a^+(k)\, a^-(k)$$

Other conserved quantities for this theory.

Problem: show that the operator $N \equiv \int d^3k\; a^+(k)\, a^-(k)$
is the Numbers operator for this theory.

$$N\,|k', k\rangle = 2\,|k', k\rangle$$

$$\frac{dN}{dt} = 0$$

for free scalar theory

there are an infinite number of conserved objects.

$$Z \equiv \int f(k_1^\mu, k_2^\mu, \ldots k_n^\lambda)\, a^+(k_1) a^-(k_2) a^+(k_3) a^-(k_4) \cdots dk_1 \ldots dk_n$$

$\frac{dZ}{dt} = 0$, Z is short for Zilch

$$(\partial^2 + m^2)\phi = 0$$

$$\langle 0 | (\partial_x^2 + m^2)\, e^{ip^\mu x_\mu} \phi(0)\, e^{-ip^\mu x_\mu} | 0 \rangle = m^2 \langle 0 | e^{ip^\mu x_\mu} \phi(0) e^{-ip^\mu x_\mu} | 0 \rangle$$

$$= m^2 \langle 0 | \phi(0) | 0 \rangle \begin{array}{l} = 0 \\ \neq 0 \end{array}$$

$\langle 0 | \phi(0) | 0 \rangle = \eta \rightarrow$ a number

if

$m^2 \eta = 0 \Rightarrow m^2 \neq 0 \Rightarrow \eta = 0$

Massive particles $\Rightarrow \langle 0 | \phi(0) | 0 \rangle = 0$

$\Rightarrow m^2 = 0$, it is possible to have $\langle 0 | \phi(x) | 0 \rangle = \eta \neq 0$

If $m^2 = 0$

~~$\Rightarrow \partial_\mu (\partial^\mu \phi(x)) = 0$~~

$m^2 = 0$

$\partial_\mu (\partial^\mu \phi(x)) = 0 \rightarrow \frac{d}{dt} Q = 0$ [if surface terms can be neglected]

$$Q = \int \partial^0 \phi(\vec{x}, t)\, d^3x$$

$$i[Q, \phi(y,t)] = i\left[\int d^3x\, \partial^0 \phi(\bar{x},t),\ \phi(y,t)\right]$$

$$= i \int d^3x\, \delta^3(x-y) = 1$$

$$i \langle 0 | [Q, \phi(y,t)] | 0 \rangle = 1$$

define: $\phi_\eta(\bar{x},t) = e^{i\eta Q}\, \phi(x,t)\, e^{-i\eta Q}$ (a)

$$= \phi(x,t) + \underbrace{i\eta[Q, \phi(\bar{x},t)]}_{\eta} + [\eta^2 Q^2, \phi]$$

$$[Q, [Q, \phi]]$$

Problem: Show terms of order η^p vanish when $p > 1$

$$\phi_\eta(x,t) = \phi(x,t) + \eta$$

if $\langle 0 | \phi(x,t) | 0 \rangle = 0$, then $\langle 0 | \phi_\eta(x,t) | 0 \rangle = \eta$

always keep $m^2 = 0$

(a) looks like a unitary transformation

Suggest that there is a new set of states

$|0_\eta\rangle = e^{i\eta Q} |0\rangle$, $|0_\eta\rangle$ is also complete.

$\Rightarrow$ $\langle b' \lambda | a' \rangle \neq 0$ for some choice of b', a'

$\langle b' | e^{-i\eta Q} | a' \rangle \neq 0$

Problem 1a) Show that $\langle 0 | 0\eta \rangle = 0$

and further that $\langle k'\eta | \ell' \rangle = 0$

b) Show $\langle 0\eta_1 | 0\eta_2 \rangle = 0,\ \eta_1 \neq \eta_2$

$$S = \int d^4x \frac{\partial^\mu \phi \partial_\mu \phi}{2} \Rightarrow \partial^2 \phi = 0$$

S invariant under $\phi(x) \to \phi(x) + \eta$

$|0\eta\rangle$ form sets of inequivalent representations of theory.

We want to deal with #'s, function - ie. not operators.

$$\left.\begin{array}{l}\langle 0 | \phi(x) | 0 \rangle \\ \langle 0 | \phi(x)\, \phi(y) | 0 \rangle\end{array}\right\} \text{Green's functions of - QFT}$$

$$\equiv \frac{1}{i} D^-(x-y)$$

do this for $(\partial^2 + m^2)\phi = 0$, $\phi(x) = \phi^-(x) + \phi^+(x)$

$$\langle 0 | \phi^-(x)\, \phi^+(y) | 0 \rangle$$

$$(\partial_x^2 + m^2) D^-(x-y) = 0$$

$$D^-(x-y) = \int d^4k\, e^{ik^\mu(x_\mu - y_\mu)} \delta(k^2 - m^2) A(k^2)\, \Theta(-k^0)$$

$$\frac{1}{i} D^-(x-y) \equiv \langle 0 | \phi(x)\, \phi(y) | 0 \rangle$$

$$\frac{1}{i} D^+(x-y) \equiv \langle 0 | \phi(y)\, \phi(x) | 0 \rangle \frac{1}{i} D^-(y-x)$$

Note: that $\langle 0|\phi(x)\,\phi(y)|0\rangle = \langle 0| e^{-ip^\mu a_\mu} e^{ip^\mu a_\mu} \phi(x) e^{-ip^\mu a_\mu} e^{ip^\mu a_\mu} \phi(y)$
$e^{-ip^\mu a_\mu} e^{ip^\mu a_\mu} |0\rangle$

$$\Rightarrow \langle 0|\phi(x)\,\phi(y)|0\rangle = \langle 0|\phi(x+a)\,\phi(y+a)|0\rangle$$

$$e^{ip^\mu(x_\mu - y_\mu)}$$

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$\phi(x)$

$\langle 0|\phi(x)|0\rangle = \langle 0|\phi(0)|0\rangle$

$\frac{1}{i} D^{-}(x-y) \equiv \langle 0|\phi(x)\phi(y)|0\rangle$

$J^{\mu\nu}|0\rangle = p^{\mu}|0\rangle = 0$, Vacuum is unchanged under Poincare trans.

Scalar field, so under LT $\phi'(x') = \phi(x)$

$\langle 0|\phi(x)\phi(y)|0\rangle = \langle 0'|\phi(x)\phi(y)|0'\rangle$, $|0'\rangle = e^{iJ^{\mu\nu}\epsilon_{\mu\nu}}|0\rangle$

$= \langle 0|\phi(x+a)\phi(y+a)|0\rangle$ → for any translationally invariant theory.

Note $|0'\rangle = |0\rangle$

$|0'\rangle = e^{ip^{\mu}a_{\mu}}|0\rangle$

$$\phi(x+a) = e^{ip^{\mu}a_{\mu}}\phi(x)\,e^{-ip^{\mu}a_{\mu}}$$

$\frac{1}{i} D(x,y) \xrightarrow[\text{Trans. invariance}]{} \equiv \frac{1}{i} D^{-}(x-y)$

$\langle 0|\phi(x)\phi(y)|0\rangle = \langle 0|e^{ip^{\mu}x_{\mu}}\phi(0)\,e^{-ip^{\mu}x_{\mu}}e^{ip^{\mu}y_{\mu}}\phi(0)\,e^{-ip^{\mu}y_{\mu}}|0\rangle$

$\langle 0|\phi(0)\,e^{-ip^{\mu}(x_{\mu}-y_{\mu})}\phi(0)|0\rangle$

$p^{\mu} \equiv$ E-Momen. Operator. (4 vector)

Introduce a complete set of states

→ $|p\rangle$

i.e ⇒ $p^{\mu}|p\rangle = p^{\mu}|p\rangle$

~~Assume~~

Assume $\sum_{\substack{\text{all possible} \\ p'}} |p'\rangle\langle p'| = I$

$I|0\rangle = |0\rangle$

$\sum_{p''} \langle 0| \phi(0) \sum_{p'} |p'\rangle\langle p'| \, e^{-ip^\mu(x_\mu - y_\mu)} \, |p''\rangle\langle p''| \, \phi(0) |0\rangle$

Assume ~~[illegible]~~ orthogonal $|p'\rangle$

$\Rightarrow \sum_{p'} \langle 0|\phi(0)|p'\rangle \, e^{-ip^\mu(x_\mu - y_\mu)} \langle p'|\phi(0)|0\rangle$

$= \sum_{p'} \left|\langle 0|\phi(0)|p'\rangle\right|^2 e^{-ip^\mu(x_\mu - y_\mu)} \, \theta(p^0)$, since $\phi^\dagger(x) = \phi(x)$

since Hamiltonian is positive-definite $\theta(p_0)$ is free to add.

~~$\int dk^2 \, \delta(k^2 - p^\mu p_\mu)$~~

$|p\rangle$, $p^\mu |p\rangle = p^\mu |p\rangle$, $\boxed{p^\mu p_\mu = p_0^2 - \vec{p}^{\,2} = k^2}$

$\int dk^2 \, \delta(k^2 - p^\mu p_\mu)$

What is $\left|\langle 0|\phi(0)|p'\rangle\right|^2$? (Replace $\sum_{p'}$ by $\int d^4p'$ in the above summation)

$\left|\langle 0|\phi(0)|p'\rangle\right|^2 = F(p^\mu p_\mu) = \int dk^2 \, \delta(k^2 - p^2) \, B(p^2)$

$\int dk^2 \int d^4p' \, e^{-ip^\mu(x_\mu - y_\mu)} \, \delta(k^2 - p^2) \, \theta(p^0) \, \beta(p^2)$

$$= \boxed{\int_0^\infty dk^2\, \beta(k^2) \int d^4p\; e^{-p^\mu(x_\mu - y_\mu)}\, \delta(k^2 - p^2)\, \Theta(p^0)} \qquad (1)$$

Any scaler field theory of form

$$S = \int d^4x\, \mathcal{L} = \int d^4x \left[\phi^\mu \partial_\mu \phi - \tfrac{1}{2} \phi^\mu \phi_\mu - \frac{m^2 \phi^2}{2} + I(\phi) \right]$$

Note: $i \langle 0 | [\phi^0(x), \phi(y)]_{x^0 = y^0} | 0 \rangle = \delta^3(x-y)$

$$i\left[\langle 0 | \phi^0(x)\phi(y) | 0 \rangle - \langle 0 | \phi(y) \phi^0(x) | 0 \rangle\right]_{x^0 = y^0} = \delta^3(x-y)$$

$$\phi^0(x) = \partial_0 \phi(x)$$

$$\left[\partial_0^x \langle \phi(x) \phi(y) \rangle\right]_{x^0 = y^0} - \left[\partial_0^x \langle \phi(y) \phi(x) \rangle\right]_{x^0 = y^0} = \delta^3(x-y)$$

Problem: Show that

$\int dk^2\, \beta(k^2)$ = constant because of the canonical commutation relations.

and find this constant.

$$\boxed{\langle \phi(x)\, \phi(y) \rangle = \int dk^2 \int \beta(k^2)\, e^{-ip^\mu (x-y)_\mu}\, \delta(p^2 - k^2)\, \Theta(p^0)\, dp}$$

This is a general result called the Lehman-Kallen result,

Problem: Show using $\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}(2k^0)^{1/2}} \left[a^{+} e - a^{-} e \right]$

$$\boxed{\frac{1}{i} \langle 0 | \phi(x) \phi(y) | 0 \rangle = \int d^4k \left[\delta(k^2 - m^2) e^{ik(x-y)} \Theta(-k^0) \right] \frac{1}{(2\pi)^3}}$$

Note: the free field and exact agree if $\boxed{\beta(K^2) = \frac{\delta(m^2)}{(2\pi)^3}}$;

$$D(x-y) \equiv i \langle 0 | [\phi(x), \phi(y)] | 0 \rangle$$

$$\left[\partial^0 D(x-y) \right]_{x^0 = y^0} = i \langle 0 | [\partial^0 \phi(x), \phi(y)] | 0 \rangle \Big|_{x^0 = y^0} = \delta^3(\vec{x} - \vec{y})$$

Trivial Problem: Show $D(x-y) = \frac{1}{(2\pi)^3 i} \int d^4k \, \epsilon(k^0) \delta(k^2 - m^2) e^{ik(x-y)}$

Problem: All these expressions have $\Theta(k^0)$, $\Theta(-k^0)$ in them. Why are they Lorentz invariant?

Hint: They always appear with a delta function $\delta(k^2 - m^2)$

10/25/2005

$$i\langle 0|\underbrace{\phi(x)\phi(y)|0\rangle \equiv D^{-}(x-y)}_{\text{free field}} = \int \frac{d^4k}{(2\pi)^3}\,\delta(k^2-m^2)\,e^{ik(x-y)}\,\theta(-k^0)$$

any interaction that respects ~~relativity and translation~~ (poincare invariance)

$$\langle 1|\phi(x)\phi(y)|1\rangle = \int dK \int \frac{d^4k}{(2\pi)^3}\,\delta(k^2-K^2)\,e^{ik(x-y)}\,\theta(-k^0)\,A(K^2)$$

$$\int_0^\infty dK^2\,A(K^2) = 1$$

For free field $A(K^2) = \delta(K^2-m^2)$

$$A(K^2) \geq 0$$

$$D(x-y) \equiv i\langle 0|[\phi(x),\phi(y)]|0\rangle$$

$$= -i\int_0^\infty dK^2 \int \frac{d^4k}{(2\pi)^3}\,A(K^2)\,\delta(k^2-K^2)\,\epsilon(k^0)$$

$$\left[\partial_0 D(x-y)\right]_{x^0=y^0} = \delta^3(\vec{x}-\vec{y})$$

$$D^{+}(x-y) \equiv i\langle 0|\phi(y)\phi(x)|0\rangle$$

$$D^{+}(x) = \frac{1}{4\pi}\,\epsilon(x^0)\,\delta(x^2) - \frac{mi}{8\pi\sqrt{x^2}}\,\theta(x^2)\left[N_1\left(m(x^2)^{1/2}\right) - i\,\epsilon(x^0)\,J_1(m\sqrt{x^2})\right]$$

Most Important 2 field Green's function

$$\langle 0|\, T(\phi(x)\phi(y))\,|0\rangle = \langle 0|\,(\phi(x)\phi(y))_+|0\rangle \equiv G(x,y)$$

$$i\langle 0|(\phi(x)\phi(y))_+|0\rangle = \Theta(x^0-y^0)\,\langle 0|\phi(x)\phi(y)|0\rangle + \Theta(y^0-x^0)\,\langle 0|\phi(y)\phi(x)|0\rangle$$

Aside: $\langle 0|\left(\phi(x_1)\phi(x_2)\phi(x_3)\ldots\phi(x_n)\right)_+|0\rangle$

Free Case: $\phi(x)$, $(\partial^2+m^2)\phi(x)=0$

$$(\partial^2+m^2)\,i\langle 0|\phi(x)\phi(y)_+|0\rangle = i\langle 0|\left(\cancel{(\partial^2+m^2)\phi(x)}\phi(y)\right)_+|0\rangle$$

$$+\, i\langle 0|\,\delta(x^0-y^0)\left[\partial^0\phi(x),\phi(y)\right]|0\rangle$$
$$+\,(\;)\,\left[\cancel{\phi(x),\phi(y)}\right]_{x^0=y^0}$$

$$=\delta(x^0-y^0)\,\delta^3(\vec{x}-\vec{y}) = \delta^4(x-y)$$

$$\left.\begin{aligned}&(\partial_x^2+m^2)\,G_0(x,y) = \delta^4(x-y)\\ &G_0(x-y) = \int\frac{d^4k}{(2\pi)^4}\, e^{ik^\mu(x_\mu-y_\mu)}\, G_0(k^2)\end{aligned}\right\}\quad \delta^4(x-y) = \int\frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}$$

$$(-k^2+m^2)\,G_0(k^2) = 1,\qquad G_0(k^2) = \frac{1}{-k^2+m^2} + c\,\delta(-k^2+m^2)$$

Fairly easy to show

$$G_0(k^2) = \frac{1}{-k^2+m^2} + \pi i \delta(-k^2+m^2)$$

$$\frac{1}{x-i\epsilon} \xrightarrow[\epsilon\to 0]{} P\frac{1}{x} + \pi i \delta(x)$$

$$G_0(x-y) = \frac{1}{-p^2+m^2-i\epsilon} \equiv P\frac{1}{-p^2+m^2} + \pi i \delta(-p^2+m^2)$$

Problems

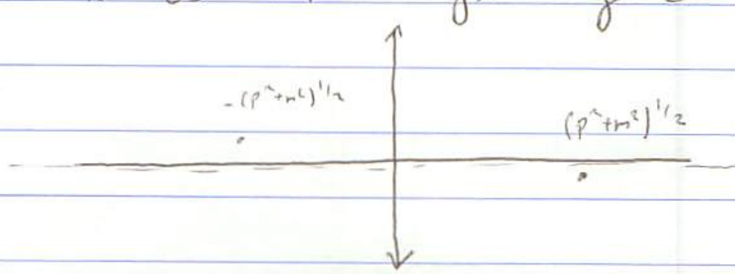
$$G_0(p^2) = \frac{1}{-p^2+m^2-i\epsilon} = P\frac{1}{-p^2+m^2} + \pi i \delta(-p^2+m^2)$$

Argue that this formula is true using calculus of residues and ~~your favorite~~ the book of your choice

$$G_0(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{1}{-p^2+m^2-i\epsilon}$$

$$\Rightarrow p^0 = \pm\sqrt{\vec{p}^{\,2}+m^2-i\epsilon}$$

It is useful to analytically continue this integral.



p_0 , $p^0 \to i p^0$

$$G_0(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{1}{-p^2+m^2-i\epsilon} \longrightarrow i \int \int_{p^4=-i\infty}^{p^4=i\infty} dp^4 d^3p \left(\frac{1}{p_4^2+\vec{p}^2+m^2-i\epsilon} \right) e^{ip(x-y)}$$

Note ~~shows~~ that if I extend x^0 into the complex plane so that $X^0 = -iX^4$

$$\left. \begin{array}{l} X^0 \to -iX^4 \\ p^0 \to ip^4 \end{array} \right\} \Longrightarrow G_0(x-y) = G_E(x^4, \vec{x}) = i \int_{-\infty}^{\infty} \frac{d^4 p_E}{(2\pi)^4} \frac{1}{(p_E^2+m^2)} e^{ip_E(x-y)}$$

$$p_E^2 = \vec{p}_4^{\,2} + \vec{p}^{\,2}$$

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$< \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)_+ > = G_0(x_1-x_2)G_0(x_3-x_4)$

+ cyclic permutations

$\mathcal{L} = \frac{(\partial_\mu \phi)^2}{2} - \frac{m^2}{2}\phi^2 + J(x)\phi(x)$

$J(x)$ is an external source

$$\mathcal{H} = \frac{(\partial^0\phi)^2}{2} + \frac{\nabla\phi^2}{2} + \frac{m^2\phi^2}{2} - J(x)\phi(x)$$

$H = \int d^3x \mathcal{H} \Rightarrow \frac{dH}{dt} \neq 0$ if $J(x)$ depends on position.

$(-\partial^2 - m^2)\phi + J(x) = 0$, $(\partial^2+m^2)\phi = J(x)$

$\phi(x) = \phi_0(x) + \left(\frac{1}{\partial^2+m^2}\right)J \rightarrow \int G_0(x-y)J(y)d^4y$

operator

notation $\langle x| G_0 J$

because

$(\partial^2+m^2)G_0(x-y) = \delta^4(x-y)$

Say

$|a'\rangle$ are a complete set of states (appropriate)

$\langle b'|\phi(x)|a'\rangle = \langle b'|\phi_0(x)|a'\rangle + G_0 J\langle b'|a'\rangle$

$$i\left(\phi(x)\phi(y)\right)_+ = i\left[\left(\phi_0 + \langle x|G_0 J\right)\left(\phi_0(y) + \langle y|G_0 J\right)\right]_+$$

$$= i\left[\phi_0(x)\phi_0(y)\right]_+ + i\left(\langle x|G_0 J\right)\phi_0(y) + i\left(\langle y|G_0 J\right)\phi_0(x)$$

$$+ i\left(\langle x|G_0 J\right)\left(\langle y|G_0 J\right)$$

$$\mathcal{G}_{ab}(\cancel{x} y) = \frac{G_{ab}(x,y)}{\langle a|b\rangle} - i\frac{\langle a|\phi(x)|b\rangle}{\langle a|b\rangle}\frac{\langle a|\phi(y)|b\rangle}{\langle a|b\rangle} = \cancel{\ldots} = i\frac{\langle a'|\left(\phi(x)\phi(y)\right)_+|b'\rangle}{\langle a'|b'\rangle}$$

$$\Downarrow$$

$$G_{ab}(x,y) \equiv i\langle a|\left[\phi(x)\phi(y)\right]_+|b\rangle$$

Problem: Show that

$$G_{ab}(x,y) = \cancel{\phi_0(x)\phi_0(y)}\ i\frac{\langle a'|\left(\phi_0(x)\phi_0(y)\right)_+|b'\rangle}{\langle a'|b'\rangle}$$

$$H = H_0^{free} - J(x)\phi(x)$$

$$\delta_J H = -\delta J(x)\phi(x)$$

$$H = \int d^3x\, \mathcal{H}, \qquad \langle b'|e^{-iHt}|a'\rangle$$

Pick $t \equiv \Delta t$ ie small

$$\cancel{\langle b'|e^{-iHt}} \quad \langle b'|e^{-iH\Delta t}|a'\rangle$$

Have complete sets of states and unitary transformations on states.

define $\langle b' \Delta t| \equiv \langle b'| e^{-iH\Delta t}$ unitary transformation.

$$\langle b'| e^{-i\Delta t \left(\int d^3x [H_0 - J(x)\phi(x)] \right)} |a'\rangle$$

$$\rightarrow \langle b'| \left(1 - i\Delta t \int d^3x [H_0 - J(x)\phi(x)] \right) |a'\rangle$$

vary J → how does this change?

$$\delta \langle b' \Delta t | a' \rangle = \cancel{\int d^3x\, \delta J(x)} \langle b'| i\Delta t \int d^3x\, \delta J(x)\, \phi(x) |a'\rangle$$

$$\cong \langle b' \Delta t | i\Delta t \int d^3x\, \delta J(x)\, \phi(x) |a'\rangle$$

$$S = \int [\overbrace{\phi^0 \partial^0 \phi - H}^{\mathcal{L}}] d^4x$$

$$\delta_J S = \int \delta J \phi\, d^4x$$

$\Rightarrow$ ~~$\delta \langle b' \Delta t | a' \rangle = \langle b' \Delta t | i\, \delta_J S | a' \rangle$~~

$$\boxed{\delta \langle b' \Delta t | a' \rangle = \langle b' \Delta t | i\, \delta \left[\int_t^{t+dt} dt \int d^3x\, \mathcal{L} \right] |a'\rangle}$$ $\Rightarrow$ Schwinger action principle

a' Δt Δt

$\langle d' \Delta t|$ $\langle e' 2\Delta t|$

$$\langle b' t | a' \rangle = \underset{\text{Sdcex}}{\sum} \langle b' t | x'(t - \Delta t) \rangle \langle x'(t - \Delta t)| - \text{——} \langle c' \Delta t | a' \rangle$$

$$\delta_J \langle b't | a' \rangle = \cancel{\ } \langle b't | \, i \, \delta_J \left[\int_0^t dt \int d^3x \, \mathcal{L} \right] | a' \rangle$$

$$\delta(\alpha\beta\gamma) = \cancel{\ } (\delta\alpha)\beta\gamma + \alpha(\delta\beta)\gamma + \alpha\beta(\delta\gamma)$$

$$\cancel{\ } \quad \delta_J \langle b't^2 | a't' \rangle = \langle b't^2 | \, i \, \delta_J \int_{t_1}^{t_2} \int d^3x \, \mathcal{L} \, | a't' \rangle$$

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$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - \frac{m^2 \phi^2}{2} + J(x)\phi(x) - \mathcal{H}_I(x)$$

$$[\, (\partial^2 + m^2)\phi(x) = J(x) \,]$$ source only

$$H_I(\phi) = \frac{\lambda}{3}\phi^3(x) + \frac{g}{4}\phi^4(x)$$

$$\boxed{(\partial^2 + m^2)\phi(x) + \lambda \phi^2(x) = J(x)}$$ cubic interaction

$|a'\rangle$ complete set of states for interacting theory

for a minute assume $J = 0$

$\langle b' | e^{iHt} | a' \rangle$

in particular investigate $t_2 - t_1$ and $t_2 \to t_1$

$$\langle b' | e^{iH\Delta t} | a' \rangle = \langle b' | 1 + iH\Delta t | a' \rangle + O(\Delta t)^2$$

assume that $H = (H_0 + H_I) - \int J(x)\phi(x)\, d^3x$

Note: This depends on time

$J(x) = j(x) + \delta J(x)$

$\delta J(x)$ is small by assumption

$$\boxed{\delta \langle b' | e^{iH\Delta t} | a' \rangle = \langle b' | \int d^3x\, \delta J(x)\, \phi(x) | a' \rangle\, i\Delta t}$$

$\Rightarrow \quad \delta \; \cancel{\langle b' | e^{iH\Delta t} | a' \rangle = \langle b' | \phi(x)}$

$$\boxed{\delta \langle b' | e^{iH\Delta t} | a' \rangle = \langle b' | \int d^4x' \, \delta J(x) \, \phi(x) \, | a' \rangle \, i}$$

$$\hookrightarrow \quad \frac{\delta \langle b' | e^{iH\Delta t} | a' \rangle}{\delta^4 J(x)} = i \langle b' | \, \phi(x) \, | a' \rangle$$

~~$\langle b' | e^{iH\Delta t} | a' \rangle$~~

<u>definition</u>: define states in the following way.

first if $J = 0$, define $| a' t_1 \rangle = e^{iHt_1} | a' \rangle$

[implicit is that H is a constant of motion],

$\langle b' t_2 | = \langle b' | e^{-iHt_2}$

$$\langle b' t_2 | a' t_1 \rangle = \langle b' | \, e^{-iH(t_2 - t_1)} \, | a' \rangle$$

t_1 |–|–|–|–|–|–| t_2

$\Delta t \qquad \Delta t$

$$\langle b' t_2 | a' t_1 \rangle = \langle b' t_2 | c'_j (t_2 - \Delta t) \rangle \langle c'_j (t_2 - \Delta t) | d'_k (t_2 - 2\Delta t) \rangle \langle d'_k (t_2 - 2\Delta t) |$$
$$\cdots \cdots \langle z'_j \, t_1 + \Delta t | a'_j t_1 \rangle$$

$$\delta_J \langle b' t_2 | a' t_1 \rangle = i \langle b' t_2 | \int \delta J(x) \, \phi(x) \, d^4x \, | a' t_1 \rangle$$

$$\hookrightarrow \quad \boxed{-i \frac{\delta}{\delta^4 J(x)} \langle b' t_2 | a' t_1 \rangle_J = \langle b' t^2 | \, \phi(x) \, | a' t_1 \rangle_J}$$

Problem Show consistent with this definition

$$\frac{\delta J(x)}{\delta J(y)} = \delta^4(x-y)$$

$$S = \int d^4x \left\{ \mathcal{L} + J(x)\phi(x) \right\}$$

$$\delta \langle b' t_2 | a' t_1 \rangle = i \langle b' t_2 | \delta S | a' t_1 \rangle \;,\quad \cancel{\delta S = \int d^4x}$$

$$= i \langle b' t_2 | \delta \int_{\text{space } t_1}^{\text{space } t_2} d^4x\, \mathcal{L} \,| a' t_1 \rangle$$

Schwinger action principle.

$$-i \frac{\delta}{\delta J(y)} \left[\langle b' t_2 | \underset{t_2 > x^0 > t_1}{\phi(x)} | a' t_1 \rangle_J \right] = ?$$

$$i\delta_J \left[\langle b' t_2 | c', t_2 - \Delta t \rangle \langle c', t_2 - \Delta t | \cdots \rangle \langle f', x^0 | \phi(x) | g', x_0 \rangle \langle a', x_0 | \cdots | a', t_1 \rangle \right]$$

$$-i \frac{\delta}{\delta J(y)} \langle b', t_2 | \phi(x) | a', t_1 \rangle = \langle b' t_2 | (\phi(y)\, \phi(x))_+ | a' t' \rangle$$

Problem From this argument show that

$$\langle b' t_2 | a' t_1 \rangle_J = \langle b' t_2, J=0 | T\left[e^{i \int_{t_1}^{t_2} d^4x\, J(x)\phi(x)} \right] | a' t_1, J=0 \rangle$$

Very special matrix element

Consider the vacuum state, $|0\rangle$

Put on source

$$|0\,t_1\rangle = T\left(e^{\int_0^{t_1}\int d^3x\, dx^0\, J(x)\phi(x)}\right)|0\rangle$$

$$\langle 0\,t_2|0\,t_1\rangle_J\Big|_{\substack{t_2\to\infty\\ t_1\to-\infty}} \equiv \langle 0^+|0^-\rangle_J \equiv Z$$

Here we assume

$$H = H_0 - \int J\phi\, d^3x$$

Note. $\frac{-i\,\delta}{\delta J(x)}\langle 0^+|0^-\rangle_J = \langle 0^+|\,\phi(x)\,|0^-\rangle$

$$\Rightarrow \frac{-i\,\delta}{\delta J(y)}\langle 0^+|\,\phi(x)\,|0^-\rangle = \langle 0^+|\left(\phi(y)\,\phi(x)\right)_+|0^-\rangle$$

$$(\partial^2+m^2)\,\phi(x) = J(x)$$

$$(\partial^2+m^2)\langle 0^+|\,\phi(x)\,|0^-\rangle = J(x)\,\langle 0^+|0^-\rangle$$

$$(\partial^2+m^2)\,\frac{\langle 0^+|\,\phi(x)\,|0^-\rangle}{\langle 0^+|0^-\rangle} = J(x)$$

$$\Phi_J(x) = \frac{\langle 0^+|\,\phi(x)\,|0^-\rangle}{\langle 0^+|0^-\rangle}\,, \quad \text{~~Note,}~~ \cancel{\phi(x) = \frac{-i\,\delta \ln Z}{\delta J(x)}}$$

Note

$$\Phi(x) = i\frac{\delta}{\delta J(x)} \ln(Z) = \frac{-i\delta}{\delta y} \ln \langle 0^+|0^-\rangle = \frac{\langle 0^+|\phi(x)|0^-\rangle}{\langle 0^+|0^-\rangle}$$

$$(\partial^2 + m^2)\, \Phi_J(x) = J$$

$$\frac{\delta}{\delta J(y)}\left[(\partial_x^2 + m^2)\, \Phi_J(x) = J\right] \Rightarrow [\partial_x^2 + m^2] \underbrace{\frac{\delta}{\delta J(y)} \Phi_J(x)}_{= G(x,y)} = \delta^4(x-y)$$

$$G(x,y) = i\left[-i \frac{\delta}{\delta J(y)} \frac{\langle \phi(x)\rangle_J}{\langle 1\rangle_J}\right]$$

$$= i\left[\frac{\langle (\phi(x)\phi(y))_+\rangle}{\langle 1\rangle} - \frac{\langle \phi(x)\rangle}{\langle 1\rangle}\frac{\langle \phi(y)\rangle}{\langle 1\rangle}\right]$$

$$(\partial^2 + m^2)\, G_J(x,y) = \delta^4(x-y)$$

$$\cancel{(\partial^2+m^2)\, G(x,y)} \quad (\partial^2 + m^2)\, G_J(x, y, x_1, x_2 \ldots x_n) = 0$$

$$\boxed{\left(\frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)}\right) G_J(x,y) = G_J(x, y, x_1, \ldots x_n)}$$ definition

10/3/2005

$$\mathcal{L} = \frac{(\partial_\mu \phi)(\partial^\mu \phi)}{2} - \frac{m^2 \phi^2}{2} + J(x)\phi(x) - g\, H_I(\phi)$$

$$e^{iHt_1} |a'\rangle = |a', t_1\rangle$$

$\vdots$

$$\left[\langle b' t_2 | a' t_1 \rangle\right]_J = \langle b' | T \left[e^{i\int_{t_1}^{t_2} dt \int d^3x\, J(\vec{x},t)\phi(\vec{x},t)} \right] |a'\rangle_{J=0}$$

for states

$$\delta_J \langle b' t_2 | a' t_1 \rangle_J = \langle b' t_2 |\, i \iint_{t_1}^{t_2} d^4x\, J(x)\, \phi(x)\, | a' t_1 \rangle$$

$$-i \frac{\delta}{\delta J(x)} \langle b' t_2 | a' t_1 \rangle_J = \langle b' t_2 | \phi(x) | a' t_1 \rangle, \quad [t_2 \leq x^0 \leq t_1]$$

else zero

$$-i \frac{\delta}{\delta J(y)} \langle b' t_2 | \phi(x) | a t_1 \rangle_J = \langle b' t_2 | (\phi(y)\phi(x))_+ | a' t_1 \rangle_J$$

Problem: Show that ~~&~~ for $\mathcal{L}$ with $g \neq 0$, $J(x)$

$$\frac{d}{dg} \langle b' t_2 | a' t_1 \rangle_{g,J} = -i \langle b' t_2 | \iint_{t_1}^{t_2} d^4x\, H_I(\phi) | a' t_1 \rangle_{g,J}$$

$$\boxed{\delta \langle b' t_2 | a' t_1 \rangle = i \langle b' t_2 | \delta \int_{t_1}^{t_2} d^4x\, \mathcal{L}\, | a' t_1 \rangle}$$

Schwinger's Action Principle } General result that condenses all results above

$$\cancel{H_I(\phi) = \frac{ig\,\phi^3(x)}{3} + \frac{\lambda\,\phi^4(x)}{4}}$$

$$\langle \phi^3 \rangle = \left(\frac{-i\delta}{\delta J}\right)^3 \langle \;\rangle$$

$$\Rightarrow \frac{d}{dg}\langle b' t_2 | a' t_1 \rangle_{J,g} = \left[-i \iint_{t_1}^{t_2} d^4x\, H_I\left(\frac{-i\delta}{\delta J(x)}\right)\right] \langle b' t_2 | a' t_1 \rangle_{J,g}$$

$$\boxed{\langle b' t_2 | a' t_1 \rangle_{J,g} = \left[e^{-ig\iint d^4x\, H_I\left(\frac{-i\delta}{\delta J}\right)}\right] \langle b' t_2 | a' t_1 \rangle_{J,g=0}}$$

This solution is incomplete

general solution is

$$\boxed{\langle b' t_2 | a' t_1 \rangle_{J,g} = e^{-i(g-g_0)\iint d^4x\, H_I\left(\frac{-i\delta}{\delta J}\right)} \langle b' t_2 | a' t_1 \rangle_{J,g_0}}$$

Simplest field theory we haven't solved yet

$$\mathcal{L} = \frac{(\partial_\mu \phi)(\partial^\mu \phi)}{2} - \frac{m^2\phi^2}{2} - \frac{g\phi^3}{3} + J\phi$$

examine this in Euclidean space $\partial^2 = -\partial^{4\,2} - \nabla^2$, $\partial^4 = i\partial^0$

and in Euclidean space

$$\frac{\delta}{\delta J(x)}\langle \;\rangle = \langle \phi(x) \rangle$$

$$\text{E.o.m} \Rightarrow -\partial^2\phi - m^2\phi - g\phi^2 + J = 0$$

dimensional destruction → go to zero space-time dimensions

$$m^2\phi + g\phi^2 = J$$

again $m^2\phi + g\phi^2 = J$

$\rangle \equiv |c\rangle =$ state of lowest energy

$\Rightarrow m^2\langle\phi\rangle + g\langle\phi^2\rangle = J\langle 1\rangle$, $\langle 1\rangle = Z$

$$m^2\frac{d}{dJ}\langle 1\rangle + g\left(\frac{d}{dJ}\right)^2\langle 1\rangle = J\langle 1\rangle$$

$$\boxed{g\left(\frac{d}{dJ}\right)^2 Z + m^2\left(\frac{d}{dJ}\right)Z = J\,Z}$$ (Airy equation)

Note: if $g=0$, $Z = \left(e^{1/m^2 J^2}\right)$ constant

Note: If we use formula from before

$$Z_{g,J} = \left[e^{g\left(\frac{d}{dJ}\right)^3}\right] Z_{g=0\,J} = e^{\left(g\frac{d}{dJ}\right)^3} e^{\frac{J^2}{m^2}} \cdot \text{constant}$$

This is one answer

differential equation has two answers.

$$Z = e^{-m^2/2g\left(\frac{J}{g^{1/2}} + \frac{m^4}{4g}\right)}\left[c\,A_i\left(\frac{J + \frac{m^2}{4g}}{g^{1/2}}\right) + d B_i(\text{same arg.})\right]$$

Airy function

back to 4 dimensions

Problem: Show that $\langle 0^+ | 0^- \rangle_{g,J} = \langle 0^+ | T\left[e^{-ig\int d^4x\, H_I(\phi)} \right] | 0^- \rangle_{J,g=0}$

Path integral

Fact

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}\alpha x^2} = \left(\frac{2\pi}{\alpha}\right)^{1/2}$$

This can be analytically continued and works for any complex a if $\int$ is defined.

(Check Sidney Coleman's lecture notes on the web)

Extend this

$$(X| = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ | \\ X_n \end{pmatrix}, \quad \text{Then} \quad \int d^n x\, e^{-\frac{1}{2}(X^T A X)} = (2\pi)^{n/2} (\det A)^{-1/2}$$

assume A is real, symmetric matrix with positive definite eigenvalues.

Notation: $[dx] = \dfrac{d^n x}{(2\pi)^{n/2}}$

$$\int [dx]\, e^{-\frac{1}{2} x A^T x} = (\det A)^{-1/2}$$

Let $Q(x) = \frac{1}{2} x^T A x + b^T x + c$

$\bar{x} \equiv$ minimum of $Q(x) \equiv -A^{-1} b$

$$Q(x) = Q(\bar{x}) + \frac{1}{2}(x-\bar{x})^T A (x-\bar{x})$$

$$Q(\bar{x}) = -\frac{1}{2} b^T A^{-1} b + c$$

$$\boxed{\int [dx]\, e^{-Q(x)} = e^{-Q(\bar{x})} (\det A)^{-1/2}}$$

$$\int [dx]\, P(x)\, e^{-Q(x)} = P\left(-\frac{\partial}{\partial b}\right) \int [dx]\, e^{-Q(x)}$$

polynomial in x

Let $n \to \infty$

$$\sum_{i=1}^{\infty} \underbrace{\phi(x_i)}_{X_i} \phi(x_i) \to \int dt\, \phi(x) \phi(x)$$

Question: What is the value of

$$\int [d\Phi]\, e^{-\frac{1}{2}\int d^4x\, \phi(x) A(x,y) \phi(y)} = (\det A)^{-1/2}$$

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$$\langle 0^+ | 0^- \rangle = Z$$

$$\delta \langle t_1 | t_2 \rangle = \langle \phi_1 | \delta \int_{t_1}^{t_2} dt \int d^3x \, \mathcal{L} \, | \phi_2 \rangle$$ → Schwinger's Action Principle

$$\mathcal{L} = \mathcal{L}_0(\phi) - g \, \mathcal{H}_I(\phi) + J(x)\phi(x)$$

$$\frac{-i\delta}{\delta J(x)} \langle 0^+ | 0^- \rangle = \langle 0^+ | \phi(x) | 0^- \rangle$$

$$\frac{-i\delta}{\delta J(y)} \langle 0^+ | \phi(x) | 0^- \rangle_{J,g} = \langle 0^+ | \big(\phi(y)\,\phi(x)\big)_+ | 0^- \rangle$$

$$_{J,g}\langle 0^+ | 0^- \rangle_{J,g} = {}_g\langle 0^+ | T \, e^{i\int d^4x \, J(x)\phi(x)} | 0^- \rangle_g$$

$$-i\frac{d}{dg} \, {}_{J,g}\langle 0^+ | 0^- \rangle_{J,g} = - {}_{J,g}\langle 0^+ | \int d^4x \, \mathcal{H}_I(\phi) \, | 0^- \rangle_{J,g}$$

(1) $$\langle 0^+ | 0^- \rangle_{J,g} = {}_J\langle 0^+ | \left(e^{-ig\int d^4x \, \mathcal{H}_I(\phi)}\right)_+ | 0^- \rangle_J = e^{-ig\int d^4x \, \mathcal{H}_I\left(\frac{-i\delta}{\delta J(x)}\right)} \langle 0^+ | 0^- \rangle_J$$

We talked about

$$\mathcal{L} = \frac{(\partial^\mu \phi)(\partial_\mu \phi)}{2} - \frac{m^2 \phi^2}{2} + J\phi$$

$$\delta\phi \Rightarrow \left(-\partial_x^2 - m^2\right)\phi(x) + J(x) = 0$$

$$(\partial_x^2 + m^2)\,\phi(x) = J(x)$$

$$\boxed{(\partial_x^2 + m^2) \frac{\langle 0^+ | \phi(x) | 0^- \rangle}{\langle 0^+ | 0^- \rangle} = J(x)}$$

$$\frac{\langle \phi(x) \rangle}{\langle 1 \rangle} = \frac{-i\,\delta}{\delta J(x)} \ln \langle 1 \rangle$$

$$(\partial_x^2 + m^2)\, G_0(x-y) = \delta^4(x-y)$$

$$G_0(x-y) = \frac{i \langle (\phi(x)\, \phi(y)) \rangle_{J=0}}{\langle 1 \rangle_{J=0}}$$

$$(\partial_x^2 + m^2)\left(\frac{-i\,\delta}{\delta J(x)} \ln Z\right) = J(x)$$

$$\frac{-i\,\delta}{\delta J(y)} \ln Z = \int (\partial_x^2 + m^2)^{-1} J(x)\, d^4x \qquad \left(\left[(\partial_x^2 + m^2)^{-1}_{xy} = G_0(x-y) \right] \right.$$

$$\boxed{\ln Z = \frac{i}{2} \int d^4y\, d^4x\; J(y)\; G_0(y-x)\, J(x)}$$

↳ Solution to $(\partial_x^2 + m^2) \dfrac{\langle \phi(x) \rangle}{\langle 1 \rangle} = J(x)$

Note: from this $\dfrac{-i\,\delta(\ln Z)}{\delta J(x)} = \int d^4y\; G_0(x-y)\, J(y)$

$$(\partial_x^2 + m^2)\left(\frac{-i\,\delta}{\delta J(x)} \ln Z\right) = (\partial_x^2 + m^2) \int d^4y\; G_0(x-y)\, J(y)$$

$$= \int d^4y\; \delta^4(x-y)\, J(y) = J(x)$$

$$\Rightarrow \langle 1 \rangle_J = \langle 0^+ | 0^- \rangle_J = Z = e^{\frac{i}{2}\int d^4x \frac{J(x)\; G_0(x-y) J(y)}{2} d^4y} \qquad (2)$$

From (1) & (2)

$$\boxed{\langle 1 \rangle_{g,J} = \left[e^{-ig\int d^4x\, \mathcal{L}_I\left(\frac{-i\delta}{\delta J_x}\right)} \right] \left[e^{\frac{i}{2}\iint d^4w\, d^4y\, \frac{J(w)\, G(w-y)\, J(y)}{2}} \right]}$$

↳ solution for any scalar field theory.

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2} = \left(\frac{2\pi}{a}\right)^{1/2}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ | \\ x_n \end{pmatrix}, \quad A \text{ is an } n\times n \text{ matrix with } A^{-1}$$

$$\int d^n x \; e^{-\frac{1}{2} x^T A x} = (2\pi)^{n/2} [\det A]^{-1/2}$$

$$d^n x \equiv dx_1\, dx_2 \ldots dx_n$$

$$[dx] \equiv \frac{d^n x}{(2\pi)^{n/2}}, \qquad Q(x) \equiv \frac{1}{2} X^T A X + b^T X + C$$

$$\int [dx]\, e^{-Q(x)} = e^{-Q(\bar{x})} (\det A)^{-1/2}$$

$$Q(\bar{x}) = \frac{-1}{2} b^T A^{-1} b + c$$

~~$P(x) \equiv$ any polynomial in x,~~

$P(x)$ = any poly. in x

$$\int [dx]\, P(x)\, e^{-Q(x)} = P\left(-\frac{\partial}{\partial b}\right) \int [dx]\, e^{-Q(x)}$$

$y_1 \to \phi_1$ | $\phi(y)$

$y_2 \to \phi_2$ Look at one dimension initially

|

$y_n \to \phi_n$

$\leftarrow a \rightarrow$ $\leftarrow a \rightarrow$

1 2 3 n

$\Rightarrow \phi(0) \to \phi_1, \quad \phi(a) \to \phi_2, \quad \phi(2a) \Rightarrow \phi_3$

Look at the integral

$$\int \underbrace{\frac{d\phi_1\, d\phi_2\, d\phi_3 \ldots d\phi_n}{(2\pi)^{n/2}}}_{[d\phi]} e^{-\frac{1}{2} a \sum_{i=1}^{n} \phi_i^2} \approx \int [d\phi]\, e^{-\int_{-\infty}^{+\infty} dt\, \phi^2(t)}$$

$$\frac{1}{2} \sum_i a\, \phi_i^2 = \frac{1}{2} \int_{-\infty}^{+\infty} dt\, \phi(t)^2$$

$$\int [d\phi]\, e^{-\frac{1}{2}\int d^4x\, \phi(x) A(x,y) \phi(y)\, d^4y} = (\det A)^{-1/2}$$

Problem: Show how you can absorb the lattice spacing in the path integral

Generalization:

$$\int [d\Phi]\, e^{i\int d^4x \left[\frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{m^2\phi^2}{2} + J\phi\right]}$$

· instead of $\frac{(\partial^\mu\phi)(\partial_\mu\phi)}{2}$, change to $\frac{1}{2}\phi(-\partial^2)\phi$, will change the action up to constants

$$\frac{1}{2}\phi(-\partial^2)\phi = \int d^4y\, \frac{1}{2}\phi(x)\left[-\partial_x^2\delta^4(x-y)\,\phi(y)\right)$$

$$\phi(x) \equiv \langle x|\Phi$$

$$-\partial_x^2\delta^4(x-y) \equiv \langle x|-\partial^2|y\rangle$$

$\Rightarrow$ Action is $i\int d^4x\, d^4y\, \frac{1}{2}\phi(x)\left[-\partial_x^2\delta^4(x-y)\right]\phi(y) - i\int \frac{m^2}{2}\phi^2(x) + J(x)\phi(x)\, d^4x$

$$\Rightarrow A = i(\partial_x^2 + m^2)\delta(x-y)$$

$$b = -J(x), \quad \frac{1}{\partial^2 + m^2} \equiv G_0$$

$$\Rightarrow e^{\frac{i}{2}\iint J(x)\, G_0(x-y)\, J(y)\, d^4x\, d^4y}$$

Read Sidney Coleman

Erice lecture notes

11/10/2005

Little review

$$\mathcal{L} = \mathcal{L}_0 + g H_I(\phi) + J(x)\phi(x)$$

$$\left(\frac{\partial^\mu\phi\partial_\mu\phi - m^2\phi^2}{2}\right)$$

source term

for $g=0$

$$|a',t\rangle_{J=0} \to e^{iHt_1}|a'\rangle_{J=0}$$

$$\frac{-i\delta}{\delta J(x)}\langle b't_2|a't_1\rangle = \langle b't_2|\phi(x)|a't_1\rangle \quad t_1 \leq x^0 \leq t_2$$

$$|0^-\rangle \to \lim_{t'\to -\infty}|0t'\rangle$$

$$\langle 0^+| = \langle 0t^2|_{t^2\to\infty}$$

$$\frac{-i\delta\langle 0^+|0^-\rangle_J}{\delta J(x)} = \langle 0^+|\phi(x)|0^-\rangle_J$$

$$\frac{-i\delta}{\delta J(y)}\frac{-i\delta}{\delta J(x)}\langle 0^+|0^-\rangle_J = -i\frac{\delta}{\delta J(y)}\langle\phi(x)\rangle = \langle(\phi(y)\phi(x))_+\rangle$$

$$(\partial^2+m^2)\phi(x) = J(x) \quad , \quad \frac{(\partial^2+m^2)\langle\phi(x)\rangle}{\langle 1\rangle} = J(x)$$

$$(\partial^2_x+m^2)G_0(x,y) = \delta^4(x-y)$$

$$\int d^4z\, G_0^{-1}(x,z)\, G_0(z,y) = \langle x|1|y\rangle$$

$$\langle x|1|y\rangle = \delta^4(x-y)$$

$$\langle x|G_0^{-1}|z\rangle\langle z|G|y\rangle \quad , \quad \langle z|G_0|y\rangle = G_0(z-y)$$

$$\langle x| G_0^{-1} |z\rangle = (\partial_x^2 + m^2)\, \delta^4(x-z)$$

$$\Rightarrow \int d^4z\, (\partial_x^2 + m^2)\, \delta^4(x-z)\, G(z-y) = (\partial^2 + m^2) G_0(x-y) = \delta^4(x-y) \checkmark$$

$$G_0^{-1} G_0 = 1$$

Now, define $\Phi_J(x) = \dfrac{\langle \phi(x)\rangle}{\langle 1\rangle}$, $\quad (\partial_x^2 + m^2)\, \Phi_J(x) = J(x)$

$$G_0^{-1} \Phi_J = J \qquad , \quad \begin{aligned} \langle x| J &= J(x) \\ \langle x| \Phi_J &= \Phi_J(x) \end{aligned}$$

$$\langle x| G_0^{-1} \Phi_J = \langle x| J$$

$$\langle x| G_0^{-1} |y\rangle \langle y| \Phi_J = \langle x| J \quad \Rightarrow \int d^4y\, (\partial_x^2 + m^2)\, \delta^4(x-y)\, \Phi_J(y) = J(x)$$

$$G_0^{-1} \Phi_J = J$$

$$\Phi_J = G_0 J \quad \Rightarrow \quad \Phi_J(x) = \int G_0(x-y)\, J(y)\, d^4y$$

$$\hookrightarrow \; G_0(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{-p^2+m^2}\, e^{ip(x-y)}$$

$$\Phi_J(x) = G_0(x-y)\, J(y)$$

$\langle 1\rangle \equiv Z \quad , \quad \ln\langle 1\rangle \equiv iW$

$$\Phi_J(x) = \frac{\langle \phi(x)\rangle}{\langle 1\rangle} = \frac{-i\delta}{\delta J(x)} \ln Z = \frac{\frac{-i\delta}{\delta J(x)} Z}{Z}$$

$$= \frac{\langle \phi(x)\rangle}{\langle 1\rangle}$$

$$\phi_J(x) = G_0(x-y)\,J(y) \Rightarrow \frac{-i\delta}{\delta J(x)}(\ln Z) = G_0(x-y)\,J(y)$$

$$\ln Z = \text{const} + \frac{i}{2}\int d^4x\, d^4y\; J(x)\,G_0(x-y)\,J(y)$$

$$=$$

$$Z = \text{const}\; e^{\frac{i}{2}\int d^4x\, d^4y\; J(x)\,G_0(x-y)\,J(y)} = \text{const}\; e^{\frac{i}{2} J^T G_0 J}$$

$$i\left(-\frac{i\delta}{\delta J(y)}\right)\left(-\frac{i\delta}{\delta J(x)}\right)\ln Z = i\left[\frac{\langle(\phi(x)\,\phi(y))_+\rangle}{\langle 1\rangle} - \frac{\langle\phi(x)\rangle}{\langle 1\rangle}\,\frac{\langle\phi(y)\rangle}{\langle 1\rangle}\right] = G_0(x-y)$$

$$\left(\frac{-i\delta}{\delta J(x)}\right)\left(\frac{i\delta J}{\delta J(x)}\right) - \left(\frac{-i\delta}{\delta J(x_n)}\right)\ln Z = 0$$

<u>Now $g \neq 0$</u>

$$\mathcal{L} = \mathcal{L}_0 + J\phi - g\,\mathcal{H}_I(\phi)$$

$$Z_{J,g} = \langle 1\rangle_{J,g}\;,\quad -i\frac{d}{dg}\langle\;\rangle_{J,g} = -\left\langle\left[\int d^4x\, \mathcal{H}_I(\phi)\right]\right\rangle_{J,g}$$

say, $\mathcal{H}_I = \phi^n(x) \rightarrow \langle\, |\int d^4x (\phi^n(x))_+ \rangle$

$$\int d^4x\,\langle(\phi^n(x))_+\rangle = \int d^4x \left(\frac{-i}{\delta J(x)}\right)^n \langle 1\rangle$$

$$= \left(\frac{-i}{\delta J(x)}\right)^n \langle\;\rangle$$

$$\frac{d}{dg}\langle\ \rangle_{g,J} = -i\int d^4x\, H_I\left(-i\frac{\delta}{\delta x}\right) \ \langle 1\rangle_{g,J}$$

$$\langle 1\rangle_{g,J} = \left[e^{-ig\int d^4x\, H_I\left(\frac{-i\delta}{\delta J(x)}\right)}\right]\langle 1\rangle_{J,g=0}$$

$$= \langle 1|T\left[e^{-ig\int d^4x\, H_I(\phi(x))}\right]|1\rangle$$

$$\boxed{\langle 1\rangle_{g,J} = \left[e^{-ig\int d^4x\, H_I\left(\frac{-i\delta}{\delta J(x)}\right)}\right] e^{\frac{i}{2}\int d^4y\, d^4z\ J(y)\, G_0(y-z)\, J(z)}}$$

$$\boxed{S = e^{\left[-\frac{1}{2} J \frac{1}{i\partial^2+m^2} J\right]}\left[\det\ i\left(\partial_x^2+m^2\right)\delta(x-y)\right]^{-1/2}}$$

$$\boxed{S = \text{const}\ \ e^{\frac{i}{2} JGJ}}$$

Feynman path integsall

$$\int [d\phi]\, e^{i\int d^4x\left[\mathcal{L}_0(\phi)+J\phi\right]} = c\, e^{\frac{i}{2} J G_0 J}$$

$$= c\langle 1\rangle$$

11/15/2005

$\mathcal{L} = \mathcal{L}_0 - g H_I + J\phi$

$$\langle 0^+|0^-\rangle_{g,J} = Z_0 \left[e^{-ig\int H_I\left(\frac{-i\delta}{\delta J(x)}\right) d^4x} \right] e^{\int d^4y\, d^4z \frac{J(y)\, G_0(y-z)\, J(z)}{2}}$$

$\overset{|||}{Z}$

$$G_0(y-z) = \frac{1}{(2\pi)^4} \int e^{ik(y-z)} \frac{1\, d^4k}{-k^2+m^2-i\epsilon}$$

For $g=0$

$$i\left[\frac{\langle \phi(x)\phi(y)\rangle_J}{\langle 1\rangle_J} - \frac{\langle\phi(x)\rangle}{\langle 1\rangle_J}\frac{\langle\phi(y)\rangle_J}{\langle 1\rangle_J} \right] = G_0(x-y)$$

$$\int [d\phi]\, e^{i\int d^4x(\mathcal{L}_0(\phi)+J(x)\phi(x))} = \text{const.}\; e^{\frac{i}{2}\int d^4y\, d^4z\, J(y) \frac{G_0(y,z)}{2} J(z)}$$

~~$\langle 0^-|0^+\rangle_{g,J} = \ldots$~~

$$\langle 0^+|0^-\rangle_{g,J} = \text{Cons.}\; e^{-ig\int d^4x\, H_I\left(\frac{-i\delta}{\delta J}\right)} \int [d\phi]\, e^{i\int d^4x\, [\mathcal{L}_0(\phi)+J\phi]}$$

$$= \text{cons.} \int [d\phi]\, e^{\underbrace{i\int d^4x\, [\mathcal{L}_0(\phi)+J(\phi)-g H_I(\phi)]}_{S(x)}}$$

$$0 = \int [d\phi] \frac{\delta}{\delta\phi(z)} e^{i\int d^4x \mathcal{L}}, \qquad \mathcal{L} = \frac{\phi(-\partial^2-m^2)\phi}{2} - g f(_3(\phi)) + J\phi$$

↳ since a total derivative
endpoints oscillate

$$0 = \int [d\phi]\, i\left[(-\partial_z^2-m^2)\left(\frac{-i\delta}{\delta J(z)}\right) + J(z) - g\left(\frac{-i\delta}{\delta J(z)}\right)^2\right] e^{i\int d^4x \mathcal{L}}$$

↑
assume $\frac{\phi^3}{3}$ interaction

$$\Rightarrow \quad 0 = \left\{ \left(-\partial^2 - m^2 \right) \left(\frac{-i\delta}{\delta J(z)} \right) + J(z) - g \left(\frac{-i\delta}{\delta J(z)} \right)^2 \right\} \int [d\phi] \, e^{i\int d^4 L}$$

$$\underset{\langle 0^+ | 0^- \rangle \equiv Z}{\parallel}$$

This is the field equation!

$\Rightarrow \quad \mathcal{L} = \mathcal{L}_0 - g H_I + J\phi$

vary $\rightarrow \quad (-\partial^2 - m^2)\phi - g\phi^2 + J = 0$

$(-\partial^2 - m^2)\langle\phi\rangle - g\langle\phi^2\rangle + J\langle 1\rangle = 0$

$\rightarrow (-\partial^2 - m^2)\left(\frac{-i\delta}{\delta J}\right)\cancel{\langle 1 \rangle} - g\left(\frac{-i\delta}{\delta J}\right)^2\langle 1\rangle + J\langle 1\rangle = 0$

~~Spin-zero (0,0) rep. of~~

Back to Fermions

Spin zero $(0,0)$ rep. of L.G

We considered $\phi^\dagger(x) = \phi(x)$

Now consider $\mathcal{L}_0(\phi_1, \phi_2) = \mathcal{L}_0(\phi_1) + \mathcal{L}_0(\phi_2) = \frac{1}{2}\left(\phi_1(-\partial^2 + m_1^2)\phi_1\right) + \frac{1}{2}\left(\phi_2(-\partial^2 + m_2^2)\phi_2\right)$

$|a_1 a_2\rangle = a_1^\dagger(k_1)\, a_1^\dagger(k_2) \_\_\_ \; a_2^\dagger(k_1)\, a_2^\dagger(k_2) \_\_ \; a_2^\dagger(k_n) |0\rangle$

if $\underline{m_1 = m_2 = m}$ $\rightarrow$ theory has a symmetry

$\Phi \equiv \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$, $\quad \mathcal{L}$ becomes $\Phi^T(-\partial^2 + m^2)\,\Phi$

define $\quad \Phi' = e^{i\sigma_2 \theta}\,\Phi \quad$ (rotation)

$\Phi' = e^{i\sigma_2\theta}\Phi$, $\Phi'^T = \Phi^T e^{-i\sigma_2\theta}$

$\Phi^T(-\partial^2 - m^2)\,\Phi = \Phi'^T(-\partial^2 - m^2)\Phi'$ ✓

Look for conserved current!

for small rotation

$\delta\Phi = i\sigma^2\,\delta w\,\Phi$

Problem: Using Noether's theorem show that $\boxed{i\,\partial^\mu\Phi\,\sigma_2\,\Phi \equiv J^\mu}$ and $\boxed{\partial_\mu J^\mu = 0}$

$\partial_\mu J^\mu = \partial_\mu\, i\left[\partial^\mu\Phi\,\sigma_2\,\Phi\right] = i\left[\underbrace{(\partial^2\Phi)^T\sigma_2\,\Phi}_{-m^2\phi\,\sigma_2\,\phi \to 0} + (\partial^\mu\Phi)^T\sigma_2\,\partial_\mu\Phi\right] = 0$

sym. times anti.

$\Rightarrow \partial_\mu J^\mu = 0 \Rightarrow \boxed{\int d^3x\; i\partial^0\Phi^T\sigma_2\Phi = Q \;,\; \frac{dQ}{dt} = 0}$

~~$\mathcal{L} = \ldots$~~

~~What if~~ What if field anticommute? no mass term ($\phi^2 = 0$)

$\mathcal{L} = \phi^\mu\partial_\mu\phi - \frac{1}{2}\phi^\mu\phi_\mu - \frac{m^2}{2}\cancel{\phi^2} \Rightarrow \delta\phi^\mu \to \partial_\mu\phi = 0$

$\delta\phi \to \partial_\mu\phi^\mu = 0$

Spin-statistics theorem: spin 0, no anticommutation.

$$Q = i\int d^3x\, \partial^0 \Phi^T \sigma_2 \Phi$$

$$[Q, \phi] = \sigma_2 \Phi$$

$$i[\partial^0 \Phi^T(x), \Phi(y)]\Big|_{x^0=y^0} = \delta^3(x-y)\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

Complex fields

define $\Theta = \dfrac{\phi_1 + i\Phi_2}{\sqrt{2}}$, $\Theta^+ = \dfrac{\Phi_1 - i\phi_2}{\sqrt{2}}$

$$\mathcal{L}_0 \to \Theta^+(-\partial^2 - m^2)\Theta$$ not matrices

$$J^\mu = i(\partial^\mu \Theta^+)\Theta - i\Theta^+\partial^\mu \Theta$$

Problem 1) Write the commutation relations in terms of Θ^+, Θ

2) Show that

$$Q \equiv \int d^3x\, J^0, \quad i[Q, \Theta] = i\Theta, \quad i[Q, \Theta^*] = -i\Theta^+$$

$$Q[\Theta|0\rangle] = [Q,\Theta]|0\rangle = Q[\Theta|0\rangle] = 1[\Theta|0\rangle]$$

Postulate $Q|0\rangle = 0$

$$Q[\Theta^+|0\rangle] = [Q, \Theta^+]|0\rangle = -1[\Theta^+|0\rangle]$$

11/17/20

$L = 1 + i\,\delta w_{\mu\nu} J_{\mu\nu}$

$\delta w_{ke} \rightarrow \delta w$ small change

$\delta w_{0\ell} \rightarrow$ Boosts

$J_{ke} \rightarrow \vec{J}$

$J_{0k} \rightarrow \vec{N}$

$J_k^{\pm} \equiv \frac{1}{2}\left(J_k \pm i N_k\right) \qquad \left(J^{\pm}\right)^{+} = J^{\mp}$ (Not Hermitian)

$[J_i^{\pm}, J_j^{\pm}] = i\,\varepsilon_{ijk} J_k^{\pm}$, $[J^{+}, J^{-}] = 0$.

$\Rightarrow$ Representations of Lorentz Group can be labeled as $\rightarrow (\ell_+, \ell_-)$

Column Matrices $\rightarrow \chi^{J^+, J^-}$

Spin 0 $\rightarrow \chi^{00} \rightarrow$ one component matrix $\rightarrow \phi(x)$

In general reps of LG transform under LT.

$L = \left\{ e^{i(\vec{w} + i\vec{\nu})\cdot J^-} \right\} e^{i(\vec{w} - i\vec{\nu})\cdot J^+} \}$

Note under LT $\phi'(x') = L\,\phi(x)$, $\phi(x') = \phi(x)$

$\chi^{00} \rightarrow \phi(x)$

$\chi^{\frac{1}{2}0}, \chi^{0\frac{1}{2}} \rightarrow$ Spin-$\frac{1}{2}$ representations of Lorentz Group.

$j = j^+ + j^-$

Notes

$\chi^{\frac{1}{2}\,0}$, $L = e^{i(\vec{w} - i\vec{v})\vec{\sigma}}$

I need to write an action

spin 0 $S = \int d^4x \{p\dot{q} - H(p,q)\}$

$$S = \int d^4x [\phi^\mu \partial_\mu \phi - \tfrac{1}{2}\phi^\mu\phi_\mu - \tfrac{m^2}{2}\phi^2]$$

$$\chi^{(\frac{1}{2},0)} \equiv \chi_+(x) = \begin{pmatrix} \chi_+(x)_1 \\ \chi_+(x)_2 \end{pmatrix}$$

$$\chi^{(0,\frac{1}{2})} \equiv \chi_-(x) = \begin{pmatrix} \chi_-(x)_1 \\ \chi_-(x)_2 \end{pmatrix}$$

under LT, $'\chi_\pm(x) = L_\pm \chi_\pm(x)$

$\underbrace{{}^T\chi^+\chi^+}_{\text{is this inv under LT?}}$, $\underbrace{{}^T\chi^+\partial_\mu\chi^+}_{\text{not LT}}$, $\overbrace{{}^T\chi^+\sigma_+^\mu\partial_\mu\chi^+}^{\text{Not LT}}$

{ Every 2x2 Matrix M can be represented as $\vec{a}\cdot\vec{\sigma} + \vec{b}$, where $\vec{a}$ is an arbitrary set of 3 complex numbers and b is an arbitrary complex number }

(Lowell Brown's Quantum Field Theory)

$$L_\pm = e^{\frac{i}{2}(\vec{\omega} \mp i\vec{\nu})\cdot\vec{\sigma}}$$

$$L_\pm^\dagger = L_\mp^{-1} \quad , \quad L_\pm^T = \sigma_2 L_\pm^{-1} \sigma_2$$

$$\Downarrow$$

$$L_\pm^T \sigma_2 L_\pm = \sigma_2$$

$$\varepsilon_{\alpha\beta} \to \begin{array}{c}\varepsilon_{11} = 0 = \varepsilon_{22} \\ \varepsilon_{12} = 1 = -\varepsilon_{21}\end{array} \Rightarrow \varepsilon_{\alpha'\beta'} L_{\pm\,\alpha'\alpha} L_{\pm\,\beta'\beta} = \varepsilon_{\alpha\beta}$$

$$\det L_\pm = \pm 1$$

but by def $\det L_\pm = 1$

Combining equations: $L_\pm^* = \left(L_\mp^T\right)^{-1} = \sigma_2 L_\mp \sigma_2$

$$(\sigma_2 \chi_\pm^*)' = \sigma_2 L_\pm^* \chi_\pm^* = \underbrace{\sigma_2 \sigma_2}_{\mathbb{1}} L_\mp \sigma_2 \chi_\pm^* = L_\mp (\sigma_2 \chi_\pm^*) = (\sigma_2 \chi_\pm^*)'$$

$\sigma_2 \chi_+^*$ transforms like χ_-
and $\sigma_2 \chi_-^*$ transforms like χ_+

Now I can make scalar objects

$$S_\pm \equiv \chi_\pm^T \sigma_2 \chi_\pm \,, \quad S_\pm' = \chi_\pm^T \underbrace{L_\pm^T \sigma_2 L_\pm}_{\sigma_2} \chi_\pm \Bigg) \text{ scalar}$$

but note $\chi_+^T \sigma_2 \chi_+ = (\chi_+)_i (\sigma_2)_{ij} (\chi_+)_j \Rightarrow 0$ if
$[(\chi_+)_i(x), (\chi_+)_j(x)] = 0$

→ From now on, assume that $\chi_\pm$ anticommute with each other
Will prove that they must, later.

$$\mathcal{L}_{mass} = m_+ \left(\chi_+ \sigma_2 \chi_+ \right) + m_+^* \chi_+^* \sigma_2 \chi_+^* + m_- \chi_- \sigma_2 \chi_- + m_-^* \chi_-^* \sigma_2 \chi_-$$
$$+ m_{+-} \chi_+^* \chi_- + m_{+-}^* \chi_-^* \chi_+$$

and $M^\mu \partial_\mu \chi$, find M^μ so that this is invariant

Problem: ~~Show that~~

Define $\sigma_\pm{}^\mu = (1, \pm\sigma^k)$

Show that

$$L_\pm^\dagger \sigma_\pm^\mu L_\pm = \ell^\mu{}_{\cdot\nu} \sigma_\pm{}^\nu$$

$$(\sigma_2 \sigma_\pm{}^\mu)_{\alpha'\beta'} (L_\pm)_{\alpha'\alpha} (L_\mp)_{\beta'\beta} = \ell^\mu{}_{\cdot\nu} (\sigma_2 \sigma_\pm{}^\nu)_{\alpha\beta}$$

(χ^*)

$$V_\pm^\mu = \chi_\pm^* \sigma_\pm^\mu \chi_\pm \quad \text{is a 4 vector}$$

$$\underbrace{V_\pm^\mu}_{LT} = \ell^\mu{}_{\cdot\nu} V_\pm^\nu$$

Most general kinetic part

$$\mathcal{L}_{kin} = a_1 \chi_+^* \sigma_+^\mu i \overleftrightarrow{\partial_\mu} \chi_+ \; a_2 \chi_-^* \sigma_-^\mu i \overleftrightarrow{\partial_\mu} \chi_-$$

$$\overleftrightarrow{\partial_\mu} \equiv \overrightarrow{\partial_\mu} - \overleftarrow{\partial_\mu}$$

$$+ c \chi_+^* \sigma_+^\mu i \overleftrightarrow{\partial_\mu} \sigma_2 \chi_-^* + c^* \chi_- \sigma_2 \sigma_+^\mu i \overleftrightarrow{\partial_\mu} \chi_+$$

11/22/2005

$\vec{J}^+, \vec{J}^-$,

$(\tfrac{1}{2}, 0), \quad (0, \tfrac{1}{2})$

$\hookrightarrow \chi^+(x) \qquad \hookrightarrow \chi^-(x)$

$\chi'_\pm(x) = L_\pm \chi_\pm(x)$

$L_\pm = e^{\frac{1}{2}(\vec{\omega} \mp i\vec{\nu})\cdot\vec{\sigma}}$

$S = \int \mathcal{L}\, d^4x$,

$\Rightarrow$ most general $\mathcal{L}$ is

$\mathcal{L}_{mass} = m_+ \chi_+^T \sigma_2 \chi_+ + m_+^* \chi_+^\dagger \sigma_2 \chi_+ + m_- \_\_ + m_{+-} \chi_+ \chi_- \_\_$

~~most general~~

$\sigma_\pm^\mu \equiv (1, \pm\sigma^k)$

easy to show that $L_\pm^\dagger \sigma_\pm^\mu L_\pm = \ell^\mu{}_\nu \sigma_\pm^\nu$

$\mathcal{L}_{kin} = a_1 \chi_+^* \sigma_+^\mu i \overleftrightarrow{\partial_\mu} \chi_+ + a_2 \chi_-^* \sigma_-^\mu i \overleftrightarrow{\partial_\mu} \chi_-$ + cross terms

$\overleftrightarrow{\partial_\mu} = -\overleftarrow{\partial_\mu} + \overrightarrow{\partial_\mu}$

$S = \int d^4x \left(\mathcal{L}_{kin} + \mathcal{L}_{mass}\right)$

Simple first guess:

Just look at terms in the action that have χ_+

$$\mathcal{L} = \underbrace{\frac{1}{2}\chi_+^* \sigma_+^\mu i\overleftrightarrow{\partial_\mu}\chi_+}_{} - \frac{1}{2} m \chi_+ \sigma_2 \chi_+ - \frac{1}{2} m \chi_+^* \sigma_2 \chi_+^*$$

$\rightarrow$ could write $i\chi_+^* \sigma_+^\mu \partial_\mu \chi_+$ (surface term vanishes)

[majorana fields]

assumes $\{\delta\chi_a^\dagger(x), \chi_b^\dagger(y)\}_{x^0=y^0} = \delta\chi_a^\dagger(x)\chi_b^\dagger(y) + \chi_a^\dagger(y)\,\delta\chi_b^\dagger(x)$, $\chi_+ = \begin{pmatrix} \chi_1^+ \\ \chi_2^+ \end{pmatrix}$

$\Rightarrow \{\delta\chi(x), \chi(y)\}\big|_{x^0=y^0} = 0$, but for scalars $[\delta\phi(x), \phi(y)]_{x^0=y^0} = 0$

anticommutation, also $\{\chi_+^*, \chi_+\} = 0$, $\{\chi_+, \chi_+\} = 0$

everything anticommutes

$S = \int d^4x\, \mathcal{L}$, $\delta S = 0$

$$\delta S = \int d^4x \left[\delta\chi_+^* \sigma_+^\mu i\partial_\mu \chi_+ + \chi_+^* \sigma_+^\mu i\partial_\mu \delta\chi_+ - \frac{1}{2} m\, \delta\chi_+ \sigma_2 \chi_+ - \frac{1}{2}\chi_+ \sigma_2 \delta\chi_+\right]$$

using $i\chi_+^* \sigma_+^\mu \partial_\mu \chi_+$ for kinetic term

~~cancel~~ due to anticommutation relations

$$\cdots = \delta S = \int d^4x \left(\delta\chi_+^* [\quad] + \delta\chi [\quad]\right)$$

$\Rightarrow$ ~~E.O.M. are: $i\sigma_+^\mu \partial_\mu \chi_+ - m\sigma_2 \chi_+^* = 0$~~

~~$\left(\sigma_+^\mu \frac{1}{i}\partial_\mu \chi_+ + m\sigma_2 \chi_+^* = 0\right)$~~

Aside

Equations of motion are

$$\boxed{\begin{aligned} & i\sigma_+^\mu \partial_\mu \chi_+ - m\sigma_2 \chi_+^* = 0 \\ & \sigma^\mu \sigma_2 \frac{1}{i}\partial_\mu \chi_+^* + m\chi_+ = 0 \end{aligned}}$$

look at mass = 0

$$i\sigma_+^\mu \partial_\mu \chi_+ = 0$$

$$\chi_+(x) = \int \frac{d^4k}{(2\pi)^4} e^{ik^\mu x_\mu} \chi_+(k)$$

$$-\sigma_+^\mu k_\mu \chi_+(k) = 0$$

$$(-\sigma^0 k_0 - \sigma^\ell k_\ell)\chi_+(k) = 0$$

$$\Rightarrow \boxed{\frac{\vec{\sigma}\cdot\vec{k}}{k^0} = 1}$$ (spin points in the direction of momentum)

define a new quantity:

$$\psi = \begin{pmatrix} \chi_+ \\ \sigma_2 \chi_+^* \end{pmatrix}$$

It follows $$\boxed{\left(\alpha^\mu \frac{1}{i}\partial_\mu + \beta m\right)\psi(x) = 0}$$ $\Rightarrow$ Dirac equation

$$\underbrace{\alpha^\mu}_{(4\times 4)} = \begin{pmatrix} \sigma_+^\mu & \underbrace{0}_{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} \\ 0 & \sigma_-^\mu \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & \underbrace{1}_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \\ 1 & 0 \end{pmatrix}$$

(~~chira~~ chiralities → neutrinos are described by χ_+ or χ_-
→ in general one needs both)

some properties of these matrices.

$$\beta^2 = 1$$

$$\{\alpha^k, \beta\} = 0$$

$$\{\alpha^k, \alpha^l\} = 2\delta_{kl}$$

$$[\alpha^k, \alpha^l] = 2i\varepsilon_{klm}\sigma_m \quad , \quad \sigma_m \text{ means here } \begin{pmatrix} \sigma_m & 0 \\ 0 & \sigma_m \end{pmatrix}$$

$$\psi(x) \to \psi'(x') = L\psi(x) \quad , \quad L = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}$$

In our original notation, ~~[illegible]~~

Infinitesimall Lorentz transformation $= 1 + i\frac{\delta\omega_{\mu\nu}\sigma^{\mu\nu}}{2}$

$$\Rightarrow L = e^{\frac{i}{4}\delta\omega^{\mu\nu}\sigma_{\mu\nu}} \quad , \text{ where now } \sigma_{kl} = \varepsilon_{klm}\sigma_m$$

$$\sigma_{0k} = -i\alpha_k$$

<u>Other matrices</u>

$$\left(\beta\alpha^\mu \frac{1}{i}\partial_\mu + m\right)\psi(x) = 0$$

define $\gamma^\mu = \beta\alpha^\mu \to$ under LT, $\boxed{L^{-1}\gamma^\mu L = l^\mu{}_\nu \gamma^\nu}$

for the γ^μ $\Rightarrow$ Dirac equation.

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}} \qquad \boxed{\left(\gamma^\mu \frac{1}{i}\partial_\mu + m\right)\psi(x) = 0}$$

11/29/2005

$$\mathcal{L} = \frac{1}{2} \chi_+^{*} \sigma_+^{\mu} \overleftrightarrow{\partial_\mu} \chi_+ - \frac{1}{2} m \chi_+ \sigma_2 \chi_+ - \frac{1}{2} m^* \chi_+^{*} \sigma_2 \chi_+^{*}$$

$$\mathcal{L}^+ = \mathcal{L}$$

Variation rules:

$$\{\delta \chi_+(x), \chi_+(y)\}_{x^0 = y^0} = 0$$

$$\delta S = \delta \int d^4x\, \mathcal{L}$$

E.O.M: $\Rightarrow \sigma_+^{\mu} \frac{1}{i} \partial_\mu \chi_+ + m \sigma_2 \chi_+^{*} = 0$

$$\sigma^{\mu} \sigma_2 \frac{1}{i} \partial_\mu \chi_+^{+} + m \chi_+ = 0$$

$$\chi_+^{+} \equiv \begin{pmatrix} \chi_1^{++} \\ \chi_2^{++} \end{pmatrix}, \quad \chi_- \equiv \sigma_2 \chi_+^{+}$$

not the initial independent χ_-, define a new var.

$$\Psi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} \equiv \begin{pmatrix} \chi_+ \\ \sigma_2 \chi_+^{+} \end{pmatrix}$$

4 independent components.

$$\Rightarrow \boxed{\left[\alpha^{\mu} \frac{1}{i} \partial_{x\mu} + \beta m\right] \Psi(x) = 0}$$

$$\alpha^{\mu} = \begin{pmatrix} \sigma_-^{\mu} & 0 \\ 0 & \sigma_+^{\mu} \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

if $\sigma_+^{\mu} = (1, \sigma^{k})$

$\sigma^{\mu} = (1, -\sigma^{k})$

$$\alpha^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha^k = \begin{pmatrix} -\sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

$$\alpha^{\mu +} = \alpha^\mu, \quad \beta^+ = \beta \qquad \{\alpha^k, \beta\} = 0$$

$$\beta^2 = 1 \qquad \{\alpha^k, \alpha^\ell\} = 2\delta_{k\ell}$$

$$[\alpha^k, \alpha^\ell] = 2i\,\varepsilon_{k\ell m}\,\sigma_m \hookrightarrow \sigma_m \equiv \begin{pmatrix} \sigma_m & 0 \\ 0 & \sigma_m \end{pmatrix}$$

$$\psi(x) = \psi'(x') = L\,\psi(x), \quad L = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}$$

$$L = e^{-1/4\, w^{\mu\nu}\sigma_{\mu\nu}}, \quad \sigma_{k\ell} = \varepsilon_{k\ell m}\,\sigma_m = \frac{1}{2i}[\alpha_k, \alpha_\ell]$$

$$\sigma_{0k} = -\sigma_{k0} = -i\alpha_k$$

$$\boxed{L_\pm^\dagger = L_\mp^{-1}}, \quad \boxed{\beta L^\dagger \beta = L^{-1}}$$

$$\boxed{\beta L^+ = L^{-1}\beta}$$

$$L^+ \alpha^\mu L = \ell^\mu{}_\nu\, \alpha^\nu$$

Dirac Eq. was $\left[\alpha^\mu \frac{1}{i}\partial_\mu + \beta m\right]\psi(x) = 0$

$$\mathcal{L} = \psi^+\left[\alpha^\mu \frac{1}{i}\frac{\overleftrightarrow{\partial}_\mu}{2} + m\beta\right]\psi \xrightarrow{L.T.} \psi^{+\prime}\left(\alpha^\mu \overleftrightarrow{\partial}'_\mu + m\beta\right)\psi'$$

$$L^{-1}\beta\alpha^\mu L = \ell^\mu{}_\nu \beta\alpha^\nu \quad , \quad \beta\alpha^\mu = \gamma^\mu$$

$$\boxed{L^{-1}\gamma^\mu L = \ell^\mu{}_\nu \gamma^\nu}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

$\Rightarrow$ Dirac equation becomes $\left[\beta\gamma^\mu \frac{1}{i}\partial_\mu + \beta m\right]\psi(x) = 0$

$$D \equiv \boxed{\left[\gamma^\mu \frac{1}{i}\partial_\mu + m\right]\psi(x) = 0}$$

$$\left(\gamma^\nu \frac{1}{i}\partial_\nu - m\right)D = \left(\gamma^\nu \frac{1}{i}\partial_\nu - m\right)\left(\gamma^\mu \frac{1}{i}\partial_\mu + m\right)\psi = 0$$

$$= \left[\gamma^\nu\gamma^\mu \frac{1}{i}\partial_\nu \frac{1}{i}\partial_\mu - m^2\right]\psi = 0$$

$$= \left[-\left(\frac{\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu}{2}\right)\partial_\mu\partial_\nu - m^2\right]\psi = \left[-g^{\mu\nu}\partial_\mu\partial_\nu - m^2\right]\psi$$

$$\Rightarrow \boxed{\left[\partial^\mu\partial_\mu + m^2\right]\psi = 0}$$

Go back to $\mathcal{L} = \psi^+\beta\left(\frac{\gamma^\mu}{2i}\overleftrightarrow{\partial_\mu} + m\right)\psi \, , \quad \psi^+\beta \equiv \bar{\psi}$

$$= \bar{\psi}\left(\frac{\gamma^\mu}{2i}\overleftrightarrow{\partial_\mu} + m\right)\psi = \mathcal{L}$$

$$\sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right]$$

$$\gamma_5 \equiv \frac{-1}{4!}\,\varepsilon_{\alpha\beta\gamma\delta}\,\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta$$

$$\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3 \;,$$

$$\gamma_5^\dagger = -\gamma_5 \;,\; \gamma_5^2 = \mathbb{1}$$

$\{\gamma_5, \gamma_\mu\} = 0$ | $1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\nu}$ Comments: These are independent matrices.

$1 + 1 + 4 + 4 + 6 = 16$

This spans the 4×4 matrix set.

$$\beta = \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv \rho_1 \;,\; \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \equiv \rho_2 \;,\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \rho_3$$

$$\beta\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = \begin{pmatrix} \chi_- \\ \chi_+ \end{pmatrix} \;,\; \gamma^5 = i\rho_3 \;,\; \Rightarrow (i\gamma_5)\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = \begin{pmatrix} -\chi_+ \\ \chi_- \end{pmatrix}$$

$$\psi_+ \equiv \frac{1}{2}(1 - i\gamma_5)\psi = \begin{pmatrix} \chi_+ \\ 0 \end{pmatrix}$$

$$\psi_- \equiv \frac{1}{2}(1 + i\gamma_5)\psi = \begin{pmatrix} 0 \\ \chi_- \end{pmatrix}$$

4x4 matrix

Take U to be unitary, $U^\dagger = U^{-1}$

$$U^{-1}\psi = \psi'$$

$$\mathcal{L} = \psi^\dagger\left(\alpha^\mu \frac{1}{2i}\overleftrightarrow{\partial_\mu} + \beta m\right)\psi$$

$$\mathcal{L} = \psi^\dagger U U^{-1}\left(\alpha^\mu \frac{1}{2i}\overleftrightarrow{\partial_\mu} + \beta m\right) U U^{-1}\psi$$

$$U^{-1}\alpha^\mu U = \alpha'^\mu \qquad \Rightarrow \mathcal{L} = (\psi^\dagger U)\left[\alpha'^\mu \frac{1}{2i}\overrightarrow{\partial_\mu} + m\beta'\right](U^{-1}\psi)$$
$$U^{-1}\beta U = \beta'$$

$$\Rightarrow \{\alpha'^\mu, \alpha'^\nu\} = \{\alpha^\mu, \alpha^\nu\}$$

$\Rightarrow$ Dirac Eq. is not unique! These transformations do not effect the physical outcome.

$\Rightarrow$ There is a transformation that makes $\psi^\dagger = \psi$

$\Rightarrow$ $\boxed{\text{Majorana field } \psi^\dagger = \psi}$ & $\boxed{\left[\gamma^\mu \frac{1}{i}\partial_\mu + m\right]\psi = 0}$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

~~in this representation~~

Majorana representation (real field)
See Brown's book!

matrix	reality	symmetry
$\mathbb{1}$	R	S
$\gamma^0 \equiv \beta$	i	A
γ_5	r	A
σ_k	i	A
σ_{0k}	i	S
γ_k	i	S

Lagrangian

$$\mathcal{L} = \underset{\substack{\uparrow \\ \text{norm.}}}{\frac{1}{2}} \left(\psi \, \alpha^\mu \frac{1}{2i} \overleftrightarrow{\partial_\mu} + \beta m \right) \psi$$

, α^μ is symmetric set of matrices
β is anti-symmetric

~~ESA~~ ⇒ Next lecture ⇒ show ψ requires anticommutation relations.

↳ Look at Jackson for relativistic E&M.

11/31/2005

Majorana spinor constructed from γ_+

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \qquad \psi^\dagger = (\psi_1, \psi_2, \psi_3, \psi_4)$$

$$, \quad \psi_i^\dagger(x) = \psi_i(x)$$

$$\mathcal{L} = \frac{-1}{2}\left[\psi^\dagger \left[\frac{\beta\gamma^\mu \overleftrightarrow{\partial}_\mu}{2i} + \beta m\right]\psi\right]$$

$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, $\beta^2 = 1$, $\gamma^0 = \beta$ imaginary + anti-symmetric matrix
$\beta^\dagger = \beta$

$\Rightarrow \psi\beta\psi$ is hermitian

$$\psi(x)\beta\psi(x) = \psi_i(x)\beta_{ij}\psi_j(x) = \beta_{ij}\left[\underbrace{\frac{\psi_i\psi_j - \psi_j\psi_i}{2}}\right] \quad \text{since } \beta \text{ is antisymmetric}$$

need anticommutation relations to have
$\psi_i\psi_j + \psi_j\psi_i = c\delta_{ij}$

To get equations of motion, we look at

$\delta\int d^4x\,\mathcal{L}$, from the kinetic energy term we get $\cancel{\delta\psi}$

$\beta\gamma^0 = 1$

$$\delta\psi\frac{\beta\gamma^\mu}{2i}\overleftrightarrow{\partial}_\mu\psi + \psi\frac{\beta\gamma^\mu}{2i}\overleftrightarrow{\partial}_\mu\delta\psi \longrightarrow \frac{-1}{2i}\left[\delta\psi\overleftrightarrow{\partial_0}\psi + \psi\overleftrightarrow{\partial_0}\delta\psi\right]$$

+ space terms.

$$= \frac{-1}{2i}\left[\partial_0[\psi\delta\psi - \delta\psi\psi] + 2\delta\psi(\partial_0\psi) - 2(\partial_0\psi)\delta\psi\right]$$

No eqn. of motion if $\delta\psi$ commutes with ψ

$$S = (\pm)\int d^4x \, \frac{\psi \overleftrightarrow{\partial_0} \psi}{2i} + \text{stuff}$$

$$P = \frac{\delta \mathcal{L}}{\delta \dot\psi} = \pm i \psi$$

in QM

$$\Rightarrow [p,q] = -i \quad , \quad \{ \pm i \psi_a(\vec{x},t), \psi_b(\vec{y},t) \} = -i \delta^3(x-y)\,\delta_{ab}$$

$$\{ \psi_a(\vec{x},t), \psi_b(\vec{y},t) \} = \pm \delta_{ab} \delta^3(x-y)$$

must be (+),

$\Rightarrow$ introduce abritrary real set of functions $f_c(\vec{x})$

$$\left\{ \int d^3x \, f_a(\vec{x}) \psi_a(\vec{x},t), \int f_b(\vec{y}) \psi_b(\vec{y},t) \, d^3y \right\} = \pm \int d^3x \, f_a(x) \, \delta_{ab} \, f_b(y) \delta^3(x-y) d^3y$$

no sum

$$= \pm \delta_{ab} \int d^3x \, f_c(x) f_b(x)$$

$$= \pm \left[\int d^3x \, f_a^2 \right]$$

RHS is positive def, LHS must be positive, choose (+) sign!

(Can also see this from playing with the Hamiltonian)

$\psi \beta \psi$, Lorentz scalar

$\psi \beta \gamma^\mu \psi \to$ Lorentz vector

$\psi \beta \gamma_5 \psi \to$ pseudo scalar

$\psi \beta \sigma_{\mu\nu} \psi \to$ L. tensor of rank 2

$\psi \beta \gamma_5 \sigma_{\mu\nu} \psi \to$ L pseudotensor

$\mathcal{L} = \psi i \beta \gamma^\mu \partial_\mu \psi - \psi \beta \psi m \quad + \mathcal{L}_{int}$

$\mathcal{L}_{int}$ could be $\quad \lambda (\psi \beta \psi)^2 + \alpha\, \psi \beta \gamma^\mu \psi\, \psi \beta \gamma_\mu \psi$

$p \to n + e^+ + \nu$

$\langle (\psi(x) \psi(y) \beta)_+ \rangle$

$(-i\gamma^\mu \partial_\mu + m)\psi(x) = 0$

$\Rightarrow \langle (\psi_a(x) (\psi(y)\beta)_b)_+ \rangle \overset{\text{free field}}{\equiv} G^0_{f_{ab}}(x-y)$

$(-i\gamma^\mu \partial_\mu + m) G^0_{f_{ab}}(x-y)$

$= \delta(x-y)\,\delta_{ab}$

$T\phi(x)\phi(y) = \begin{cases} \phi(x)\phi(y) & x^0 > y^0 \\ \phi(y)\phi(x) & y^0 > x^0 \end{cases}$

$T(\psi(x)\psi(y)) = \begin{cases} \psi(x)\psi(y) & x^0 > y^0 \\ -\psi(y)\psi(x) & y^0 > x^0 \end{cases}$

$G^0_f(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik^\mu (x-y)_\mu}}{\gamma^\mu k_\mu + m - i\epsilon} = \int \frac{d^4k}{(2\pi)^4} \left[\frac{e^{ik^\mu (x-y)_\mu}}{+k^2 + m^2 - i\epsilon} \right] (\gamma^\mu k_\mu - m)$

All this was for χ_+

$\Rightarrow \quad \Psi \equiv \begin{pmatrix} \psi_{\chi+} \\ \psi_{\chi-} \end{pmatrix} = 8$ components

each component hermitian field.

$[i\gamma_s^\mu \partial_\mu - m]\Psi = 0$

This trick is needed for electric charge

$\gamma_s^\mu = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & \gamma^\mu \end{pmatrix}$

$\bar{\Psi}_s \beta_s \gamma^\mu_s \, q \, \Psi = J^\mu$ (in fact it is an electric current!)

$q \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \leftarrow$ 8x8 matrix
mixes χ_+ and χ_- components

Using the ~~Super~~ Dirac equation

$\partial_\mu J^\mu = 0$ ✓ $\Rightarrow Q = \int d^3x \, J^0$

$\frac{dQ}{dt} = 0, \quad [Q, \Psi_s] = q \Psi_s$

$q^2 = 1$, so Q when applied to states extracts positive or negative integers.

~~Super Ψ = Dirac spinor~~

<u>Define:</u> $\Psi_D = \frac{1}{\sqrt{2}} \left(\Psi_{\chi_+} + i \Psi_{\chi_-} \right)$

$\hookrightarrow$ 4 comp. complex Dirac spinor.

12/6/2005

$D(\frac{1}{2},0) \oplus D(0,\frac{1}{2})$

$$\mathcal{L} = c\,\bar{\psi}(\sigma^\mu \partial_\mu - m)\psi$$

$$\bar{\psi} = \psi^\dagger \beta \quad , \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

$D(1,0)$, $D(0,1)$

$D(\frac{1}{2},\frac{1}{2}) \rightarrow$ spin 1 and spin 0

$\searrow$

$A^\mu(\vec{x},t)$

$A^k(\vec{x},t) =$ 3 vector under rotations $\rightarrow A^k$

$A^0(\vec{x},t) =$ scalar under rotation $\rightarrow A^0 = \phi$

$A^\mu(\vec{x},t)$ under LT $\quad A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x)$

~~A~~ Write E.O.M?

$p\dot{q} - H \quad , \quad F^{\mu\nu}\partial_\mu A_\nu$

$p_\mu \dot{A}^\mu$

Write $F^{\mu\nu} = \dfrac{F^{\mu\nu} + F^{\nu\mu}}{2} + \dfrac{F^{\mu\nu} - F^{\nu\mu}}{2} = F^{\mu\nu}_{sym} + F^{\mu\nu}_{anti}$

$$\Rightarrow F^{\mu\nu}_s \partial_\mu A_\nu + F^{\mu\nu}_{anti} \partial_\mu A_\nu$$

$\parallel$

$$F^{\mu\nu}_s\left(\frac{\partial_\mu A_\nu + \partial_\nu A_\mu}{2}\right) + F^{\mu\nu}_{Anti}\left(\frac{\partial_\mu A_\nu - \partial_\nu A_\mu}{2}\right)$$

$\partial_\mu A_\nu - \partial_\nu A_\mu \to$ 6 terms

$F^{\mu\nu}_{anti} \longrightarrow$ 6 terms

look at $F^{\mu\nu}$ under spatial rotations.

$F^{0k}_{Anti} \rightarrow$ spatial 3 vector

$F^{ij}_{Anti} \longrightarrow$ spatial $\underbrace{\text{3 vector}}_{\text{pseudovector}}$

(Symmetric part has spin 2 parts, choose the antisymmetric parts)

$$\mathcal{L} = \underbrace{F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)}_{p\dot{q}}$$

What about time derivatives?

$$\tfrac{1}{2} F^{0k}(\partial_0 A_k - \partial_k A_0) + \tfrac{1}{2} F^{k0}(\partial_k A_0 - \partial_0 A_k) =$$

$$\frac{2}{2} F^{0k}(\partial_0 A_k - \partial_k A_0)$$

$p \rightarrow F^{0k}$

$\dot{q} \rightarrow \partial_0 A_k$

$$\mathcal{L} = \frac{F^{\mu\nu}}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{m^2}{2} A^\mu A_\mu + c\, F^{\mu\nu} F_{\mu\nu}$$

EM Lagrangian $\rightarrow$ note no m^2

$$-\frac{1}{2}\left[F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) - \frac{F^{\mu\nu} F_{\mu\nu}}{2}\right]$$

Field Equations $\rightarrow \delta \int d^4x \mathcal{L} \rightarrow \delta F^{\mu\nu}\left[(\partial_\mu A_\nu - \partial_\nu A_\mu) - F_{\mu\nu}\right]$

$$\delta A_\nu \left[2\partial_\mu F^{\mu\nu}\right] = 0$$

$\Rightarrow$ $\boxed{\partial_\mu F^{\mu\nu} = 0}$, $\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu}$

if there was an interaction

$$\mathcal{L} = \frac{F^{\mu\nu}}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) - \frac{F^{\mu\nu}F_{\mu\nu}}{4} + e J^\mu_{ext} A_\mu$$

$\Rightarrow$ $\boxed{\partial_\mu F^{\mu\nu} = e J^\nu_{ext}}$, $\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu}$

Note $\rightarrow$ $\underbrace{\partial_\nu \partial_\mu F^{\mu\nu}}_{0} = e\, \partial_\nu J^\nu_{ext}$

since F_{Anti} $\Rightarrow$ $\boxed{\partial_\nu J^\nu_{ext} = 0}$

Take $J^\mu_{ext} = 0$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad \partial_\mu F^{\mu\nu} = 0$$

$$\partial_\mu\left[\partial^\mu A^\nu - \partial^\nu A^\mu\right] = 0 \quad \Rightarrow \partial^2 A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0$$

$\Rightarrow$ Note if $'A^{\mu}(x) \equiv A^{\mu}(x) + \partial^{\mu}\lambda(x)$ $\equiv$ Gauge transformation.

Then since

$$\partial_\mu 'A_\nu - \partial_\nu 'A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$$

The form of the action doesn't change

$$\mathcal{L} = \frac{'F^{\mu\nu}}{2}\left(\partial_\mu 'A_\nu - \partial_\nu 'A_\mu\right) - 'F^{\mu\nu} F_{\mu\nu}$$

$$\Rightarrow \; 'F_{\mu\nu} = \partial^{\mu} 'A_\nu - \partial_\nu 'A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \quad , \quad 'F^{\mu\nu} = F^{\mu\nu}$$

because of gauge freedom you are free to replace

$$A^\mu \rightarrow A^\mu + \partial^\mu \lambda$$

Because of gauge freedom (gauge invariance) you can always find an $'A^\mu = A^\mu + \partial^\mu \lambda \;\ni\; \partial_\mu 'A^\mu = 0$ Lorentz gauge

Similarly, you can always pick $'A^\mu \;\ni\; \vec{\nabla} \cdot \vec{A} = 0$ $\Rightarrow$ Colomb (radiation) gauge

Lorentz Gauge

$$\partial^2 A^\mu = 0$$

$$\mathcal{L} = \frac{\partial_\mu A^\nu \, \partial_\nu A^\mu}{2} \; , \quad \text{has } p \text{ for } \partial_0 A^0$$

~~fails~~ quantization fails

Radiation Gauge

$$\vec{\nabla} \cdot \vec{A} = 0 \; , \quad \partial^2 A^k - \partial^k (\partial_0 A^0 + \cancel{\partial_k A^k}^{\,0}) = 0$$

$$\partial^2 A^0 - \partial^0 (\partial_0 A^0 + \cancel{\partial_k A^k}_{\,0}) = 0$$

$$\left(\partial^{0^2} - \nabla^2\right) A^0 - \partial_0^2 A^0 = 0$$

$$\Rightarrow -\nabla^2 A^0 = 0 \quad \left(\text{if current} = e J^0_{ext}\right)$$

not manifestly causal.

$$\partial^2 A^k - \partial^k (\partial_0 A^0) = 0$$

but if no current, $A^0 = 0$

$$\Rightarrow \partial^2 A^k = 0, \quad \text{but note } A^k = {}^T A^k, \text{ i.e } \nabla \cdot \vec{A} = 0$$

$$\Rightarrow A^k(x,t) = \int \frac{d^4 k}{(2\pi)^4} e^{i k^\mu x_\mu} A^k(k)\, \delta(k^2)$$

$$\Rightarrow \nabla \cdot A^k(x) = 0, \quad \vec{k} \cdot \vec{A}(k) = 0$$

$$\partial^2 {}^T A^k = 0$$

E_k

$$\boxed{[A^i(\bar{x},t), F^{0k}(\bar{x}',t)] = i\left(\delta^{ik} - \frac{\partial^i \partial^k}{\nabla^2}\right) \delta^3(x-x')}$$

Commutation relations

comes from constraint $\frac{\partial}{\partial x^i} A^i = 0$

$$[A^i, A^j]_{t'=t} = 0, \quad [F^{0k}, F^{0l}]_{t'=t} = 0$$

projection operator

Note that $\frac{\partial}{\partial x^i}\left[\delta^{ik} - \frac{\partial^i \partial^k}{\nabla^2}\right] \delta^3(x-x') = \left[\frac{\partial}{\partial x_k} - \frac{\nabla^2 \partial^k}{\nabla^2}\right] \delta = (\partial^k - \partial^k)\delta = 0$ ✓

$$i\left\langle \left[A^{\mu}(x)\, A^{\nu}(y)\right]_{+} \right\rangle_{\text{free}} = G^{\mu\nu}(x,y) = \overbrace{g^{\mu\nu}\, D(x-y)}^{\text{Feynman propagator.}}$$

propagator for free scalar field.

12/8/2005

$$\mathcal{L} = \frac{\partial^\mu\phi\,\partial_\mu\phi}{2} - \frac{m^2}{2}\phi^2 + J\phi$$

$$\to (-\partial^2 - m^2)\phi + J = 0, \quad (\partial^2+m^2)\phi = J$$

$$G_0(x-y) = \left(\frac{1}{\partial^2+m^2}\right)\delta(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{ip(x-y)}}{-p^2+m^2-i\epsilon}$$

$$i\left[\frac{\langle(\phi(x)\phi(y))_+\rangle}{\langle 1\rangle} - \frac{\langle\phi(x)\rangle}{\langle 1\rangle}\frac{\langle\phi(y)\rangle}{\langle 1\rangle}\right] = G_0(x-y)$$

$$Z = \langle 0^+|0^-\rangle_J = C\, e^{\frac{i}{2}\int d^4x\, d^4y\, J(x)\, G_0(x-y)\, J(y)} = e^{i\frac{JG_0J}{2}}$$

$$i\left[\frac{-i\delta}{\delta J(y)}\frac{-i\delta}{\delta J(x)}\right]\ln Z = G^0(x-y)$$

⇒ Now add an interaction Hamiltonian $-g\mathcal{H}_I(x)$

$$\mathcal{L} = \frac{\partial^\mu\phi\,\partial_\mu\phi}{2} - \frac{m^2}{2}\phi^2 + J\phi - g\mathcal{H}_I(x)$$

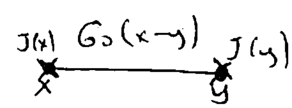
$$\boxed{Z_{g,J} = \left[e^{ig\int d^4z\, \mathcal{H}_I\left(\frac{-i\delta}{\delta z}\right)}\right] e^{i\frac{JG_0J}{2}}\cdot \text{constant}}$$

→ Discussion doesn't worry much about constants, i's etc.

$$Z_{g=0,J} \cong e^{i\frac{JGJ}{2}} = 1 + i\frac{JGJ}{2} + \left(\frac{iJGJ}{2}\right)^2\frac{1}{2} + \dots \left(\frac{iJGJ}{2}\right)^n\frac{1}{n!} + \dots$$

⇓

$$i\int_{x,y}\frac{J(x)\,G(x-y)\,J(y)}{2}\,d^4x\,d^4y \to$$



$$\Rightarrow \quad Z_{g=0,J} = 1 + \times\!\!\longrightarrow\!\!\times + \begin{matrix} \times\!\!\longrightarrow\!\!\times \\ \times\!\!\longrightarrow\!\!\times \end{matrix} + \begin{matrix} \times\!\!\longrightarrow\!\!\times \\ \times\!\!\longrightarrow\!\!\times \\ \times\!\!\longrightarrow\!\!\times \end{matrix}$$

Take $\mathcal{L}_I = \dfrac{\phi^3}{3}$, $\mathcal{L}_I \longrightarrow \left(\dfrac{-i\,\delta}{\delta J(z)}\right)^3$

$$\cancel{Z_{g,J} = \left[1 + \left(\frac{\delta}{\delta J(z)}\right)^3\right]}$$

$$Z_{g,J} = \left[1 + g\int\left(\frac{\delta}{\delta J(z)}\right)^3 d^4z + \frac{g^2}{2}\left[\int\left(\frac{\delta}{\delta J(z)}\right)^3\right]^2 + \text{———}\right]\left[1 + \times\!\!\longrightarrow\!\!\times + \begin{matrix} \times\!\!\longrightarrow\!\!\times \\ \times\!\!\longrightarrow\!\!\times \end{matrix} + \text{——}\right)$$

$$Z_{g,J} \overset{O(g^0)}{=} 1 + \times\!\!\longrightarrow\!\!\times + \begin{matrix} \times\!\!\longrightarrow\!\!\times \\ \times\!\!\longrightarrow\!\!\times \end{matrix} \text{------}$$

$$\frac{\delta}{\delta J(z)} Z_{g=0,J} = \langle \phi(z) \rangle_{J,g=0}$$

$$\langle \phi(z) \rangle_{J,g=0} = \frac{\delta}{\delta J(z)} \underbrace{\int d^4x\, d^4y \frac{J(x)\, G_0(x-y)\, J(y)}{2}} + \frac{\delta}{\delta J(z)}\left[\begin{matrix} \times - \times \\ \times - \times \end{matrix}\right]$$

$$= \int \frac{d^4x\, d^4y}{2}\left[\delta^4(x-z)\, G_0(x-y)\, J(y) + \delta^4(y-z)\, G_0(x-y)\, J(x)\right]$$

$$\langle \phi(z) \rangle_{J,g=0} = \int d^4y\, G_0(z-y)\, J(y) + \frac{\delta}{\delta J(z)}\left[\begin{matrix} \times - \times \\ \times - \times \end{matrix} + \text{———}\right]$$

using short-hand notation

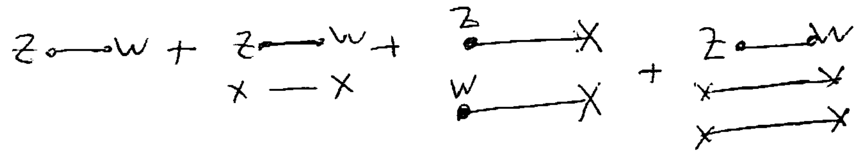
$$\langle \phi(z) \rangle_{J,g=0} = \underset{z}{\bullet}\!\!\longrightarrow\!\!\times + 2 \begin{matrix} \overset{z}{\bullet}\!\!\longrightarrow\!\!\times \\ \times\!\!\longrightarrow\!\!\times \end{matrix} + \begin{matrix} \overset{z}{\bullet}\!\!\longrightarrow\!\!\times \\ \times\!\!\longrightarrow\!\!\times \\ \times\!\!\longrightarrow\!\!\times \end{matrix}$$

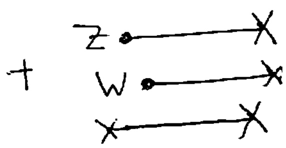
Note

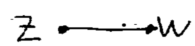
$$\langle \phi(z) \rangle_{g=0,\,J=0} = 0 \quad \checkmark \qquad \text{(Does not include symmetry breaking)}$$

$$\langle (\phi(z)\,\phi(w))_{+} \rangle_{g=0,\,J=0} = \left[\frac{\delta}{\delta J(w)} \langle \phi(z) \rangle_{g=0,\,J} \right]_{J=0} =$$

$$\frac{\delta}{\delta J(w)} \langle \phi(z) \rangle_{g=0,\,J} =$$





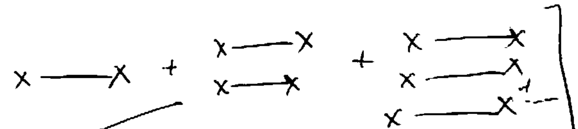
Note: $\quad \dfrac{\delta}{\delta J(z)} \dfrac{\delta}{\delta J(w)} Z_{g=0,\,J=0} =$ 

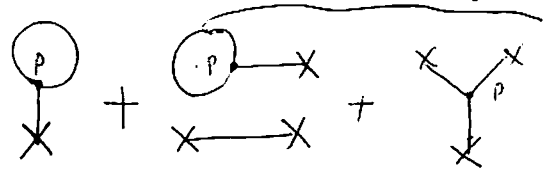
$$\qquad\qquad\qquad\qquad /\!/\!/ \qquad G_0(z-w)$$

$\Longrightarrow$ Back to cubic intesaction $\dfrac{\phi^3}{3}$

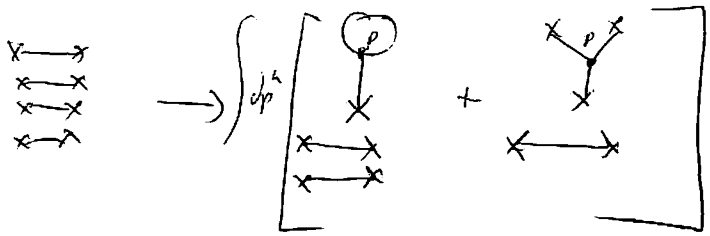
$$Z_{g,J} \overset{O(g^0)}{=} Z_{g=0,\,J}$$

$$Z_{g,J} \overset{O(g')}{=} g \int \left(\frac{\delta}{\delta J(p)} \right)^3 d^4 p \;\; Z_{g=0,\,J} \;,$$

$$Z_{g'} = \int \left(\frac{\delta}{\delta J} \right)^3 \left[1 + \right.$$  $$\left. \right] d^4 p$$

$$= \int d^4 p \; \Big[$$ 

for higher terms, these graphs repeat with added ×—×



$$\langle \phi \rangle \Big|_{J=0} \overset{O(g^1)}{=} \frac{\delta}{\delta J(m)} Z_{g^1} \Big|_{J=0} =$$

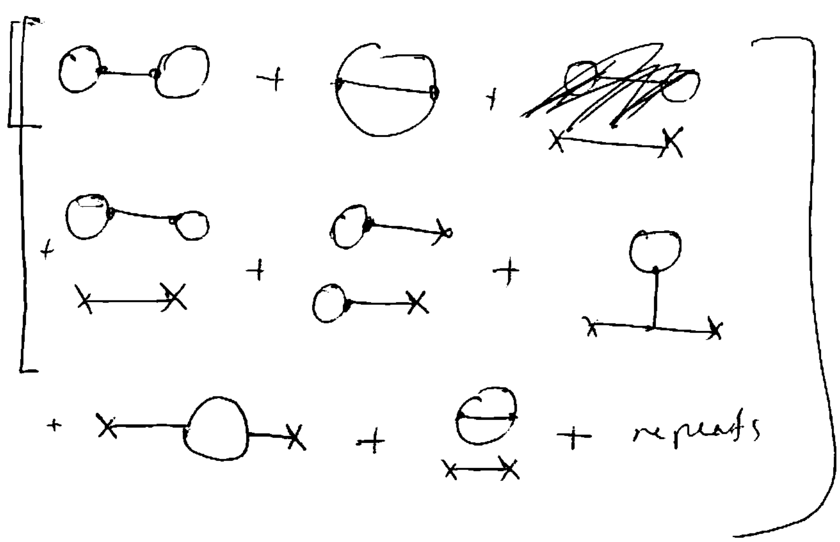
$$\int G_0(m-p)\, G_0(p-p)\, d^4p = \left[\int d^4_p\, G_0(m-p) \right] G_0(0)$$

$$= \left[\int d^4_p \int \frac{d^4k}{(2\pi)^4} e^{ik^\mu (m-p)_\mu} \left(\frac{1}{-k^2+m^2-i\epsilon} \right) \right] \underbrace{\left[\int d^4\alpha \frac{1}{-\alpha^2+m^2-i\epsilon} \right]}_{\text{as } \alpha^2 \to \infty}$$

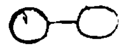
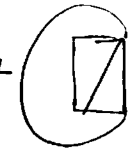
↳ this integral goes to ∞

$O(g^2)$,

$$Z_{g^2,J} = g^2 \Big[\quad \Big]$$

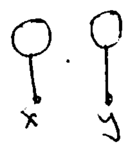
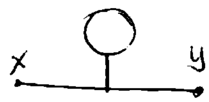


+ repeats

$$\langle 1 \rangle_{g,J=0} = 1 + \Big[$$  $$+$$  $$\Big] g^2 + g^4 \Big[$$  $$+$$  $$\Big)$$

$$\langle \phi \rangle = g$$  $$+ g^3$$  - - - - - -

$$\langle (\phi(x)\,\phi(y)) \rangle_{+g^2} =$$  $$+$$ 

$$+$$  $$+$$  $$+ O(g^4)$$